POINTWISE CONVERGENCE OVER FRACTALS FOR DISPERSIVE EQUATIONS WITH HOMOGENEOUS SYMBOL

DANIEL ECEIZABARRENA AND FELIPE PONCE-VANEGAS

Abstract. We study the problem of pointwise convergence for equations of the type $i\hbar\partial_t u + P(D)u = 0$, where the symbol P is real, homogeneous and non-singular. We prove that for initial data $f \in H^s(\mathbb{R}^n)$ with $s > (n-\alpha+1)/2$ the solution u converges to f $\mathcal{H}^\alpha$-a.e, where $\mathcal{H}^\alpha$ is the α-dimensional Hausdorff measure. We improve upon this result depending on the dispersive strength of the symbol. On the other hand, we prove negative results for a wide family of polynomial symbols P. Given α, we exploit a Talbot-like effect to construct regular initial data whose solutions u diverge in sets of Hausdorff dimension α. However, for quadratic symbols like the saddle, other kind of examples show that our positive results are sometimes best possible. To compute the dimension of the sets of divergence we use a Mass Transference Principle from Diophantine approximation theory.

1. Introduction

We study the problem of pointwise convergence to the initial datum for the following dispersive partial differential equations:

$$\begin{aligned} i\hbar\partial_t u + P(D)u = 0, \\ u(\cdot,0) = f \in H^s(\mathbb{R}^n), \end{aligned} \tag{1}$$

where $\hbar = 1/(2\pi)$, $D = -i\hbar\partial$, and $P \in C^\infty(\mathbb{R}^n\backslash\{0\})$ is a real, non-singular function, *i.e.* $\nabla P(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$. We will work with functions P that are homogeneous of degree $k \geq 1$, $k \in \mathbb{R}$; these symbols are also referred to as of principal type. In general, we will denote the solution to (1) by $T_t f$.

The classical problem concerns the Schrödinger equation, which corresponds to $P(\xi) = |\xi|^2$. Carleson asked in [8] for the minimal regularity $s \geq 0$ such that all functions $f \in H^s(\mathbb{R}^n)$ satisfy $\lim_{t\to 0} e^{it\Delta/\hbar} f = f$ almost everywhere. For $n = 1$, he proved that $s \geq 1/4$ is sufficient, while Dahlberg and Kenig [11] showed that this is also necessary.

For dimensions $n \geq 2$, the problem was subsequently investigated by several authors in [38, 36, 30, 24, 6, 13, 15], just to mention a few. In [7], Bourgain gave a counterexample, discussed in detail in [31], to prove that $s \geq \frac{n}{2(n+1)}$ is necessary, while Du, Guth and Li proved in [14] that $s > 1/3$ is sufficient in $n = 2$, and Du and Zhang [16] proved that $s > \frac{n}{2(n+1)}$ is sufficient for $n \geq 3$. Thus, the problem has been solved except for the endpoint.

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Several variations of the problem have been proposed; for example, convergence along curves [25, 9], convergence for other equations [3, 32, 33, 10], and convergence in other manifolds [42, 17], some papers addressing more than one version of the problem.

We consider here the fractal refinement of the problem, namely, the convergence $\mathcal{H}^\alpha$-almost everywhere, where $\mathcal{H}^\alpha$ is the α-Hausdorff measure. It has been studied, for example, in [2, 27, 26, 28]. The question is to determine the critical regularity

$$s_c(\alpha) = \inf \left\{ s \geq 0 \mid \lim_{t\to 0} T_t f = f \ \alpha\text{-a.e.} \quad \text{for every } f \in H^s(\mathbb{R}^n) \right\},$$

which has the following properties:

(i) $s_c(\alpha) \leq n/2$, because if $f \in H^s(\mathbb{R}^n)$ for some $s > n/2$, then $T_t f$ is continuous and the solution converges everywhere;
(ii) $s_c(\alpha)$ is a non-increasing function in α; and
(iii) $s_c(\alpha) \geq (n-\alpha)/2$, because for $s < (n-\alpha)/2$ a function $f \in H^s(\mathbb{R}^n)$ can diverge in a set of dimension α [39].

It is frequent to rephrase this problem in terms of almost everywhere convergence with respect to Frostman measures, that is, probability measures supported in the unit ball B_1 that satisfy $\mu(B_r) \leq C r^\alpha$ for all $r > 0$. Denoting the collection of all such measures by $M^\alpha(B_1)$, Frostman's lemma implies that the critical regularity can also be computed by

$$s_c(\alpha) = \inf \left\{ s \geq 0 \mid \lim_{t\to 0} T_t f = f \ \mu\text{-a.e.} \ \text{ for every } \mu \in M^\alpha(B_1) \text{ and } f \in H^s(\mathbb{R}^n) \right\}.$$

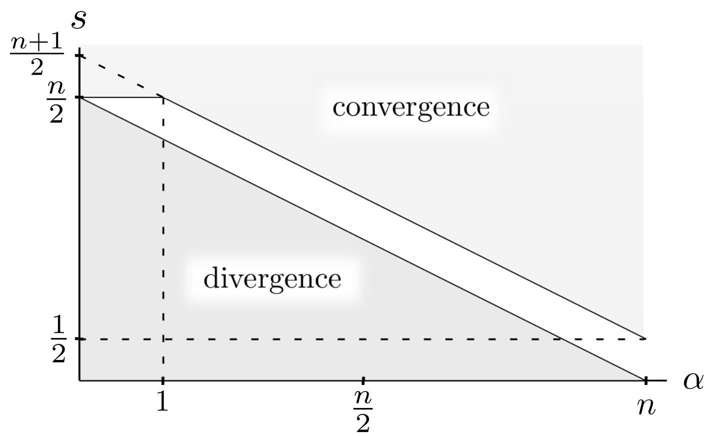
Convergence is usually better in presence of dispersion, that is, when the frequencies of the solution travel in different directions at different speeds. In these cases, the L^2-norm is constant, while the L^∞-norm decays, and the rate of decay measures the strength of the dispersion. An example is the Schrödinger equation, with $P(\xi) = |\xi|^2$, which satisfies both conditions. An example of no dispersion is the transport equation, with $P(\xi) = \xi$.

Let us first see what might happen when the solution lacks dispersion. Let $\beta > 0$. As we said, if $s < (n-\beta)/2$, a function $f \in H^s(\mathbb{R}^n)$ can diverge in a set $E \subset \mathbb{R}^n$ with Hausdorff dimension β. Since high frequencies remain concentrated, at times $t > 0$ the solution $u(\cdot, t)$ might still diverge in a large set E_t with dimension β. One may naively think, based on Cavalieri's principle (which, in the fractal setting, works in some situations but is in general false [18, Ch. 8]), that all the sets E_t make a set F of dimension $\beta + 1$ in $\mathbb{R}^n \times \mathbb{R}$. Moreover, if the sets E_t are more or less disjoint, then the projection of F to $\mathbb{R}^n$ could have dimension $\beta + 1$. Thus, in this bad scenario, the solution might diverge in a set of dimension $\alpha = \beta + 1$. In other words, if $s < (n-\alpha+1)/2$ and $f \in H^s(\mathbb{R}^n)$, we should expect the solution u to diverge in set of dimension α.

1.1. **Positive results.** Our first result is precisely $s_c(\alpha) \leq (n-\alpha+1)/2$, that is, that the situation just described is the worst case scenario. Since we expect it to hold when no dispersion is present, we call $(n-\alpha+1)/2$ the non-dispersive threshold.

Theorem 1.1 (The non-dispersive upper bound). *Let $P \in C^\infty(\mathbb{R}^n \setminus \{0\})$ be a non-singular, homogeneous function of degree $k \geq 1$, for $k \in \mathbb{R}$, and let $s > (n-\alpha+1)/2$. Then,*

$$\lim_{t\to 0} T_t f = f \quad \mu\text{-a.e.}, \qquad \forall \mu \in M^\alpha(B_1) \quad \text{and} \quad \forall f \in H^s(\mathbb{R}^n).$$



When $\alpha = n$, this result has already been proved in several cases:

- for $P = |\xi|^2$ by Vega [38] and Sjölin [36];
- for P a polynomial of degree two by Rogers *et al.* [32];
- for P a polynomial of principal type by Ben-Artzi and Devinatz [3].

For general α, this result was proved for P elliptic of degree $m \geq 2$ by Sjögren and Sjölin [35].

Let us come back to the dispersive case. When the high frequencies are dispersed in different directions at different speeds, one could think that the probability that high frequencies concentrate at many spots most of the time is small, so one expects that at a fixed point in space the solution tends to evolve somehow smoothly. Kato called attention to this fact in [23]. Therefore, we expect to improve the non-dispersive bound if the equation disperses frequencies. While this is actually the case for many symbols P, it turns out that there are symbols such as $P(\xi) = \xi_1^2 - \xi_2^2$ for which the non-dispersive bound cannot be improved, as was proved in [32] for $\alpha = n$.

Our next result shows that, in the presence of dispersion and when α is small, we can achieve the lowest possible $s_c(\alpha) = (n-\alpha)/2$. How small α can be chosen depends on the strength of the dispersion.

Theorem 1.2 (The dispersive bound)**.** *Suppose there exists $\beta > 0$ such that $\|T_t\varphi\|_\infty \leq C_\varphi |t|^{-\beta}$ for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with Fourier support in $\{|\xi| \simeq 1\}$. Then, for $\alpha < \beta$ and $s > (n-\alpha)/2$, $T_t f$ converges to f μ-a.e. for every $f \in H^s(\mathbb{R}^n)$ and for every $\mu \in M^\alpha(B_1)$.*

If D^2P is non-singular, then $\|T_t\varphi\| \leq_\infty C_\varphi |t|^{-n/2}$ for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\operatorname{supp} \widehat{\varphi} \subset \{\xi \in \mathbb{R}^n : |\xi| \simeq 1\}$; see Theorem 1 in Ch. 8.3 of [37]. As an immediate consequence, we get:

Corollary 1.3. *If $D^2P(\xi)$ is non-singular for $\xi \neq 0$ and if $\alpha < n/2$, then $s_c(\alpha) = (n-\alpha)/2$.*

This result was proved by Barceló *et al.* [2] for $P = |\xi|^2$, and they also proved it for $P = |\xi|^m$ with $m > 1$ when $\alpha < (n-1)/2$.

Theorem 1.1 and Corollary 1.3, together with the fact that $s_c(\alpha)$ is non-increasing, imply that if D^2P is non-singular, then

$$\begin{aligned} s_c(\alpha) &= (n-\alpha)/2, && \text{if } \alpha \leq n/2, \\ s_c(\alpha) &\leq n/4, && \text{if } n/2 \leq \alpha \leq n/2+1, \\ s_c(\alpha) &\leq (n-\alpha+1)/2, && \text{if } n/2+1 \leq \alpha \leq n. \end{aligned} \tag{2}$$

In even dimensions, we will see that for $P = \xi_1^2 + \cdots + \xi_{n/2}^2 - \xi_{n/2+1}^2 - \cdots - \xi_n^2$ the equalities are actually attained.

When D^2P is positive definite, we can complement Corollary 1.3 with Du and Zhang's deep result in Theorem 2.4 of [16], so that

$$\begin{aligned}
&s_c(\alpha) = (n-\alpha)/2, && \text{if } \alpha \le n/2,\\
&s_c(\alpha) \le n/4, && \text{if } n/2 \le \alpha \le (n+1)/2,\\
&s_c(\alpha) \le \frac{n}{2(n+1)} + \frac{n}{2(n+1)}(n-\alpha), && \text{if } (n+1)/2 \le \alpha \le n.
\end{aligned}$$

This is already known for the Schrödinger equation.

1.2. **Negative results.** In the second part of the paper we discuss negative results. To build counterexamples, the recurring idea is to place as many wave packets as possible in a carefully chosen plane in (x,t), in such a way that the oscillations interact coherently, making the solution to be large in a large set over the plane.

We consider symbols P of the type

$$P(\xi) = \xi_1^k + W(\xi'), \qquad \xi = (\xi_1, \xi') = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n,$$

where $k \ge 2$ is an integer and $W \in \mathbb{Q}[X_2, \dots, X_n]$ has degree k. We will impose some additional conditions to the polynomial W in Section 3. The case $P(\xi) = |\xi|^2$, corresponding to $k = 2$, was investigated in [29]; however, we draw here a more complete picture of the divergence sets involved. The case $k \ge 3$ and $\alpha = n$ was tackled by An, Chu and Pierce in [1], which is our main source of inspiration. Our main result in this direction is the following.

Theorem 1.4. *Let $P(\xi) = \xi_1^k + W(\xi')$, where $W \in \mathbb{Q}[X_2, \dots, X_n]$ is non-singular in the sense that $\nabla W(\xi') \neq 0$ for every $\xi' \in \mathbb{C}^{n-1} \setminus \{0\}$.*

Suppose that either $2 \le k \le 2(n-1)$ and

$$s < \begin{cases} (i)\ \ \dfrac{1}{4} + \dfrac{n-1}{4(n(k-1)+1)} + \dfrac{(n-1)(k-1)}{2(n(k-1)+1)}(n-\alpha) & \text{for } n - \dfrac{n-1}{2k} \le \alpha \le n,\\[2ex] (ii)\ \dfrac{1}{4} + \dfrac{n-1}{4(n+k-1)} + \dfrac{(n-1)}{2(n+k-1)}(n-\alpha) & \text{for } n - \dfrac{1}{2} - \dfrac{n-1}{k} \le \alpha \le n - \dfrac{n-1}{2k}, \end{cases}$$

or $k > 2(n-1)$ and

$$s < \begin{cases} (i)\ \ \dfrac{1}{4} + \dfrac{n-1}{4(n(k-1)+1)} + \dfrac{(n-1)(k-1)}{2(n(k-1)+1)}(n-\alpha) & \text{for } n - \dfrac{n-1}{2k} \le \alpha \le n,\\[2ex] (ii)\ \dfrac{1}{4} + \dfrac{n-1}{4(n+k-1)} + \dfrac{(n-1)}{2(n+k-1)}(n-\alpha) & \text{for } n - \dfrac{n-1}{k-n+1} \le \alpha \le n - \dfrac{n-1}{2k},\\[2ex] (iii)\ \dfrac{1}{4} + \dfrac{n-1}{4k} + \dfrac{n-1}{4k}(n-\alpha) & \text{for } n - \dfrac{k+n-1}{2k-n+1} \le \alpha \le n - \dfrac{n-1}{k-n+1}. \end{cases}$$

Then there exists $f \in H^s(\mathbb{R}^n)$ such that $T_t f$ diverges in a set of Hausdorff dimension α.

In the theorem we stated the hypothesis of non-singularity as $\nabla W(\xi') \neq 0$ for every $\xi' \in \mathbb{C}^{n-1} \setminus \{0\}$ because it is easier to read, but the theorem might hold for a larger class of polynomials. We will state the precise condition in Section 3.

The proof is based on the counterexamples designed by Bourgain [31], who exploited Gauss sums and the underlying Talbot effect to compute the solution at rational times. When $k > 2$, the oscillatory sums demand a finer manipulation, so An, Chu and Pierce [1] introduced to this context Deligne's Theorem 8.4 from [12].

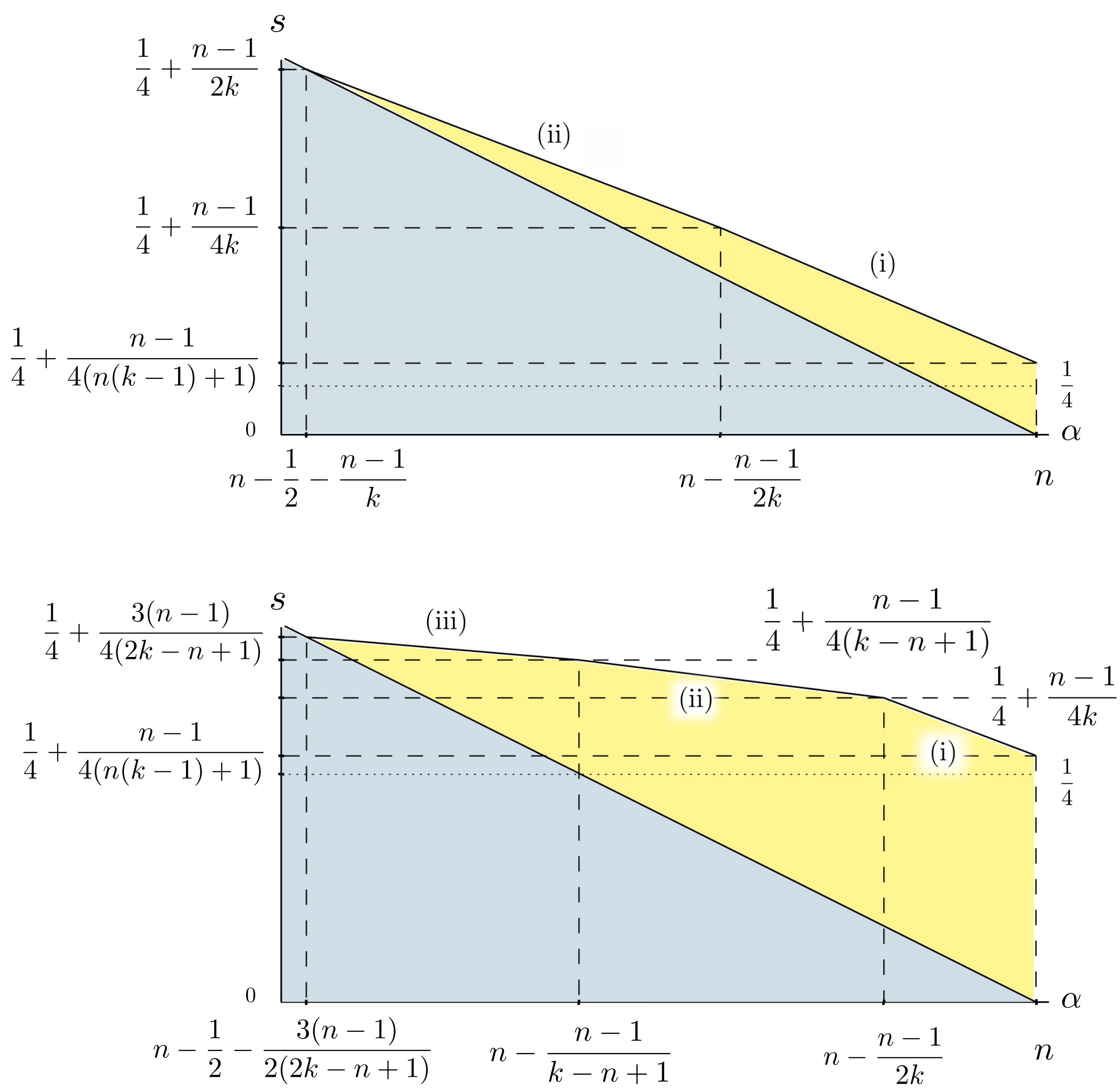


FIGURE 1. Representation of Theorem 1.4. The plot at the top is $k \le 2(n-1)$, and the plot at the bottom is $k > 2(n-1)$. In the blue region the initial datum itself diverges, so only the yellow region is not trivial. The numbering (i), (ii) and (iii) corresponds with that in the theorem.

The resulting divergence sets are very similar to those studied in diophantine approximation. To compute their Hausdorff dimension, we decided to use the mass transference principle, a powerful tool in that field introduced by Beresnevich and Velani [4]. We motivate it briefly here, and we will discuss it in more detail in Section 4.

A prototypical example of the sets considered in diophantine approximation is the limsup of sets F_m which are unions of balls $\{B_i\}$ of radius R_m^{-1}, where R_m is a sequence diverging to infinity. To compute the Hausdorff dimension of this limsup, one usually takes the union of such sets up to some scale m and sorts out the set F_m to get a large subset F_m' which is somehow uniformly distributed. Then, one applies a Frostman measure technique to compute the dimension. This can be very technical and exhausting; the reader can find an example in Section 5 of [29]. On the other hand, the mass transference principle runs the whole process automatically and efficiently:

to compute the dimension of a limsup of balls $B(x_i, r_i)$, all it requires is to find an exponent a such that the limsup of the dilated balls $B(x_i, r_i^a)$ has full measure.

The theorem of Beresnevich and Velani [4] only permits dilations that take balls to balls, so their mass transference principle is usually referred to as from balls to balls. However, our divergence sets are not union of balls, but rectangles, so this result does not directly apply. Wang, Wu and Xu [41] proved a mass transference principle from balls to rectangles, but this is still not sufficient for our purposes. Recently, Wang and Wu [40] were able to prove a version from rectangles to rectangles. This is the form we use here.

Finally, we study non-elliptic quadratic symbols. By a linear transformation, it suffices to study the polynomials

$$P(\xi) = \xi_1^2 + \cdots + \xi_m^2 - \xi_{m+1}^2 - \cdots - \xi_n^2 \qquad \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n,$$

where we assume that $1 \le m \le n/2$ without loss of generality. By (2), we only need to work with $\alpha > n/2$. For large α, we prove that the non-dispersive exponent given in Theorem 1.1 is optimal. For intermediate α, we improve the trivial $(n-\alpha)/2$ without reaching the non-dispersive threshold. Notice that Theorem 1.4 still applies here; however, we can build even more regular initial data with solutions that diverge on a set with the same dimension α. Our counterexamples are inspired by the work of Rogers, Vargas and Vega [32], who considered the case $\alpha = n$. In contrast to theirs, our results are sensitive to the index m. Results are portrayed in Figure 2.

Theorem 1.5. *Let P be a quadratic polynomial with index $1 \le m \le n/2$.*

- *If $m \le n/2 - 1$, let*

$$s < \frac{n + (n-2m)(n-\alpha)}{2(n-2m+2)} \qquad \textit{for } n/2 \le \alpha \le n-m+1,$$

$$s < (n-\alpha+1)/2 \qquad \textit{for } n-m+1 \le \alpha \le n.$$

- *If n is odd and $m = (n-1)/2$, let*

$$s < (n-\alpha+m+1)/4 \qquad \textit{for } (n+1)/2 \le \alpha \le (n+3)/2,$$

$$s < (n-\alpha+1)/2 \qquad \textit{for } (n+3)/2 \le \alpha \le n.$$

- *If n is even and $m = n/2$, let*

$$s < (n-\alpha+1)/2 \qquad \textit{for } n/2+1 \le \alpha \le n.$$

Then there exists $f \in H^s(\mathbb{R}^n)$ such that $T_t f$ diverges in a set of Hausdorff dimension α.

Theorem 1.5 together with Theorem 1.1 imply $s_c(\alpha) = (n-\alpha+1)/2$ when $\alpha \ge n-m+1$. In particular, we fully determine $s_c(\alpha)$ when n is even and $m = n/2$:

$$s_c(\alpha) = \begin{cases} (n-\alpha)/2 & \text{for } \alpha \le n/2, \\ n/4 & \text{for } n/2 \le \alpha \le n/2+1, \\ (n-\alpha+1)/2 & \text{for } n/2+1 \le \alpha \le n. \end{cases}$$

Outline of the paper.

- Positive results:

Section 2: We prove Theorems 1.1 and 1.2.

- Negative results:

Section 3: We build the counterexample for Theorem 1.4 based on the one proposed by Bourgain in [7] and An, Chu and Pierce in [1] and we determine the divergence sets.

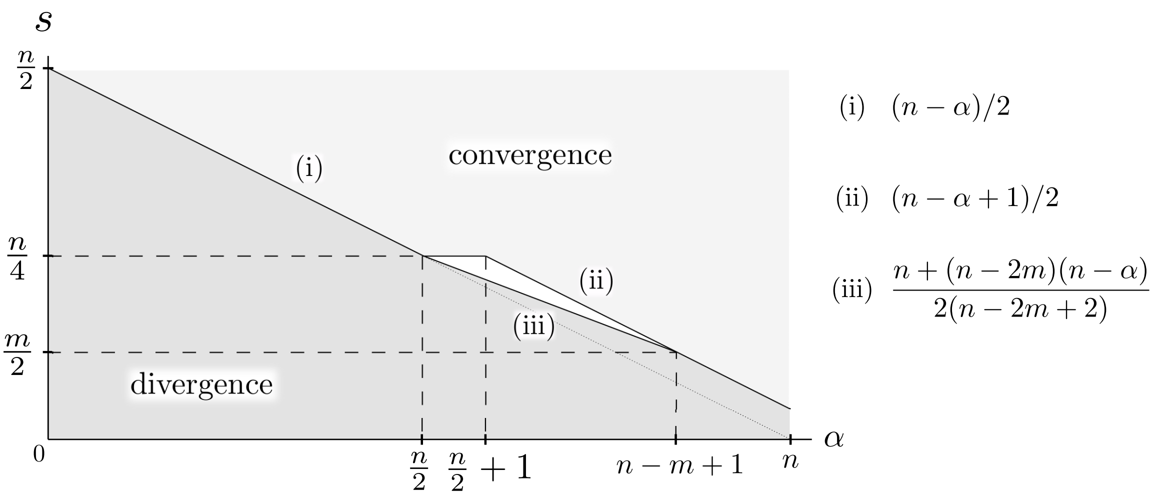


FIGURE 2. Representation of Theorem 1.5 for the case $m \leq n/2 - 1$.

Section 4: We discuss the mass transference principle, which we use to compute the Hausdorff dimension of the divergence sets.

Section 5: We study in detail the divergence sets, which depend on certain parameters. We compute their dimension. Strictly speaking, the theorem follows by studying only some extremal parameters; however, the mass transference principle allows us to deal with the whole range of parameters and ensure that this is the best possible result for this counterexample.

Section 6: Once we know the dimension of the divergence sets, we compute the Sobolev regularity of the counterexample and conclude the proof of Theorem 1.4.

Section 7: We prove Theorem 1.5.

Notations.

(a) Miscellaneous: $e(z) = e^{2\pi i z}$; $B(a,r) = \{x : |x-a| \leq r\}$.

(b) Relations: $A \lesssim B$ means that $A \leq CB$ for some constant $C > 0$; analogously, we have $A \gtrsim B$ and $A \simeq B$. When we want to stress some dependence of C on a parameter N, we write $A \lesssim_N B$. We write $c \ll 1$ as a shorthand of "a sufficiently small constant".

(c) Algebra: if k is a field, then $\bar{k} \supset k$ is the unique, up to isomorphism, algebraic closure of k. For a prime q, $\mathbb{F}_q = \mathbb{Z}/q\mathbb{Z}$. The projective space $\mathbb{P}^d(k)$ is $k^{d+1} \setminus \{0\}$ with the relation $x \sim y$ if $x = \lambda y$ for some $\lambda \in k$. If R is a ring, then $R[X_1, \ldots, X_n]$ are the polynomials with coefficients in R.

(d) Size of sets: If $E \subset \mathbb{R}^n$ is a Lebesgue measurable set, then either $|E|$ or $\mathcal{H}^n(E)$ denote its Lebesgue measure. If E is a finite set, then $|E|$ is the number of elements. For a set A, the s-Hausdorff content $\mathcal{H}^s_\delta(A)$ at scale δ is defined in (95). The s-Hausdorff measure is $\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A)$. The Hausdorff dimension is $\dim_{\mathcal{H}} A = \inf\{s \geq 0 \mid \mathcal{H}^s(A) = 0\}$.

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2. Convergence Results

In this section we prove Theorems 1.1 and 1.2. As usual, we aim to bound the maximal operator $f \mapsto \sup_{0\le t\le 1}|T_t f|$. After space-time rescaling, we localize in time like in [24] and discretize the maximal operator like in [16], which leaves us with a more manageable operator.

The solution of (1) is

$$T_t f(x) = \int \widehat{f}(\xi) e(x\cdot\xi + tP(\xi))\, d\xi.$$

Since we will restrict the frequencies of f to lie in some annulus, we can localize with a cut-off $\varphi \in C_0^\infty(\mathbb{R}^n)$ and express the solution as

$$T_t f(x) = \int \varphi(\xi)\widehat{f}(\xi) e(x\cdot\xi + tP(\xi))\, d\xi.$$

In this form, $T_t f$ is the Fourier transform of a measure over the graph of P, *i.e.* over $S := \{(\xi, P(\xi)) \mid \xi \in \operatorname{supp}\varphi\}$, so that

$$T_t f(x) = (\widehat{f}\, dS)^\vee(x,t).$$

We will adopt this point of view at times.

Since our results refer to convergence μ-a.e for measures $\mu \in M^\alpha(B_1)$, we should define $T_t f$ at least at that level of precision. If f is a measurable function, then we choose the representative

$$\widetilde{f}(x) := \lim_{r\to 0} \frac{1}{|B(x,r)|}\int_{B(x,r)} f, \tag{3}$$

whenever the limit exists. If $f \in H^s(\mathbb{R}^n)$ and $s > (n-\alpha)/2$, the limit (3) exists μ-a.e. Moreover, if we write $f = J_s * \langle D\rangle^s f$, where $\langle D\rangle^s f \in L^2(\mathbb{R}^n)$ and $\widehat{J_s}(\xi) := \langle 2\pi\xi\rangle^{-s}$ is the Bessel potential, then the integral defining the convolution is absolutely convergent μ-a.e., and $J_s * \langle D\rangle^s f = \widetilde{f}$ μ-a.e. We refer the reader to Definition 1.4 and Proposition 7.1 of [22] for details. Hence, we can write $(T_t f)^\sim$ also as $J_s * T_t\langle D\rangle^s f$.

By standard arguments, for any $s' > s$, the solution $T_t f$ converges to $f \in H^{s'}(\mathbb{R}^n)$ μ-a.e. if

$$\|\sup_{0\le t\le 1}|T_t f|\|_{L^2(\mu)} \le CR^s\|f\|_2, \quad \text{for every } R \gg 1,$$

where $\operatorname{supp}\widehat{f} \subset \{|\xi| \simeq R\}$. Applying the transformation $(x,t) \mapsto (x/R, t/R^k)$ and using the homogeneity of P, that is, that $R^{-k}P(R\xi) = P(\xi)$, the expression above is equivalent to

$$\|\sup_{0\le t\le 1}|T_t f|\|_{L^2(\mu)} = R^{-\alpha/2}\|\sup_{0\le t\le R^k}|T_t f_R|\|_{L^2(\mu_R)},$$

where $f_R(x) := f(x/R)$ and $d\mu_R(x) := R^\alpha d\mu(x/R)$. Therefore, our goal has changed to prove

$$\|\sup_{0\le t\le R^k}|T_t f_R|\|_{L^2(\mu_R)} \le CR^{s-(n-\alpha)/2}\|f_R\|_2, \tag{4}$$

where $\operatorname{supp}\widehat{f_R} \subset \{|\xi| \simeq 1\}$ and μ_R is a probability measure with support in B_R that satisfies $d\mu_R(B_r) \le Cr^\alpha$ for all $r > 0$, that is, $\mu_R \in M^\alpha(B_R)$.

We remark that, with a few modifications, it is possible to extend some arguments to non-singular, almost k-homogeneous symbols P, that is, symbols that satisfy

$$0 < c_1 \le \frac{1}{R^{k-1}}|(\nabla P)(R\xi)| \le c_0, \text{ for } |\xi| \simeq 1, \qquad \text{and} \qquad \frac{1}{R^{k-2}}|(D^2P)(R\xi)| \le c_2, \qquad \forall R \gg 1.$$

Our next step is to localize in time as Lee did in Lemma 2.3 of [24]. We show that instead of proving the maximal estimate for $t < R^k$ as in (4), it is enough to prove it for $t < R$.

Lemma 2.1. *Let P be a non-singular symbol, i.e. $\nabla P(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$. Suppose that for some $q \geq 2$ and $\mu \in M^\alpha(B_R)$ it holds that*

$$\| \sup_{0 \leq t \leq R} |T_t f| \|_{L^q(\mu)} \leq C R^\beta \|f\|_2 \tag{5}$$

for every $f \in L^2(\mathbb{R}^n)$ such that $\operatorname{supp} \widehat{f} \subset \{|\xi| \simeq 1\}$*. Then,*

$$\| \sup_{t \in \mathbb{R}} |T_t f| \|_{L^q(\mu)} \leq C R^\beta \|f\|_2, \tag{6}$$

where C depends on ∇P.

As a remark, we note that in the original argument of [24] there are ϵ-losses in the power of R. However, these losses were removed in Lemma 2.1 of [25].

Proof. We can assume that $R \geq 1$, otherwise (6) always holds. We cover the time line $\mathbb{R}$ with disjoint intervals I_l of length $|I_l| = R$ and center Rl, for $l \in \mathbb{Z}$, so that the supremum at (6) can be replaced by

$$\sup_{t \in \mathbb{R}} |T_t f(x)| \leq \sup_l \sup_{t \in I_l} |T_t f(x)| \leq \Big(\sum_l \sup_{t \in I_l} |T_t f(x)|^q \Big)^{1/q}.$$

We take the $L^q(\mu)$-norm at both sides to reach

$$\| \sup_{t \in \mathbb{R}} |T_t f| \|_{L^q(\mu)} \leq \Big(\sum_l \| \sup_{t \in I_l} |T_t f| \|^q_{L^q(\mu)} \Big)^{1/q}. \tag{7}$$

Choose a function $\psi \in \mathcal{S}(\mathbb{R})$ such that $\psi \geq 1$ in $[-1, 1]$ and $\widehat{\psi} \geq 0$ is supported in $[-5, 5]$. Choose one more function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with the same properties, and define the function of localization to I_l as

$$\psi_{R,l}(t) \varphi_R(x) := \psi(R^{-1}(t - Rl)) \varphi((c_0 R)^{-1} x).$$

The constant $c_0 \geq 1$ is just to signal that we will need to adjust the support later. We localize $T_t f$ and write it as

$$\begin{aligned}
\psi_{R,l}(t)\, \varphi_R(x)\, T_t f(x) &= \Big[(\widehat{\psi}_{R,l}\, \widehat{\varphi}_R) * (\widehat{f}\, dS) \Big]^\vee (x, t) \\
&= \int \Big[\int \widehat{\psi}_{R,l}(\tau - P(\eta))\, \widehat{\varphi}_R(\xi - \eta)\, \widehat{f}(\eta)\, d\eta \Big] e^{2\pi i (x \cdot \xi + t\tau)}\, d\xi d\tau.
\end{aligned}$$

After the change of variables $(\xi, \tau) \mapsto (\xi, P(\xi) + \tau)$ we find that

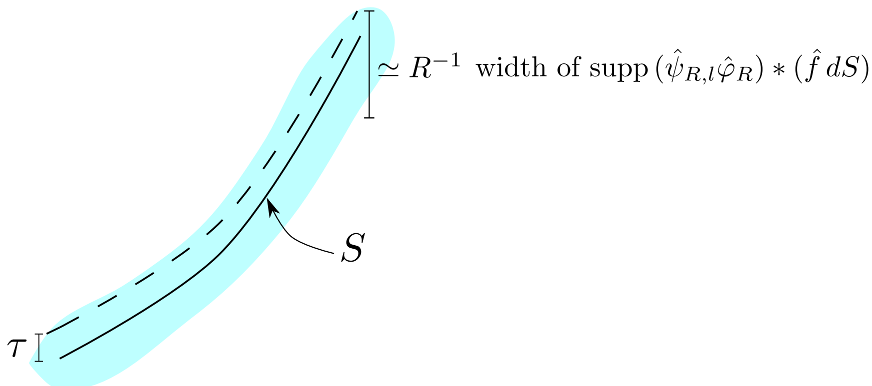
$$\begin{aligned}
&\psi_{R,l}(t)\, \varphi_R(x)\, T_t f(x) = \\
&\qquad \int \Big[\int \widehat{\psi}_{R,l}(P(\xi) - P(\eta) + \tau)\, \widehat{\varphi}_R(\xi - \eta)\, \widehat{f}(\eta)\, d\eta \Big] e^{2\pi i (x \cdot \xi + t P(\xi))}\, d\xi\, e^{2\pi i t \tau}\, d\tau.
\end{aligned} \tag{8}$$

We redefine the function enclosed by parentheses in (8) as

$$\widehat{f}_{\tau,l}(\xi) := \int \widehat{\psi}_{R,l}(P(\xi) - P(\eta) + \tau)\, \widehat{\varphi}_R(\xi - \eta)\, \widehat{f}(\eta)\, d\eta. \tag{9}$$

For $|\xi - \eta| \leq (c_0 R)^{-1}$, the function $\widehat{f}_{\tau,l}$ vanishes if $|P(\xi) - P(\eta) + \tau| \geq 5R^{-1}$. Thus, if we choose $c_0 \geq 10 \sup_{|\xi| \simeq 1} |\nabla P(\xi)|$, by the mean value theorem $\widehat{f}_{\tau,l}$ vanishes if $|\tau| > 10R^{-1}$. Consequently, it suffices to integrate (8) in $|\tau| \leq 10R^{-1}$, so rewrite (8) as

$$\begin{aligned}
\psi_{R,l}(t)\, \varphi_R(x)\, T_t f(x) &= \int_{|\tau| \leq 10R^{-1}} \widehat{f}_{\tau,l}(\xi) e(x \cdot \xi + t P(\xi))\, d\xi\, e^{2\pi i t \tau}\, d\tau, \\
&= \int_{|\tau| \leq 10R^{-1}} T_t f_{\tau,l}(x)\, e^{2\pi i t \tau}\, d\tau.
\end{aligned}$$



After this localization, let us apply the hypothesis (5) to each addend in (7) so that

$$\begin{aligned}\|\sup_{t\in I_l}|T_t f|\|_{L^q(\mu)} &\le \|\sup_{t\in I_l}|\psi_{R,l}(t)\varphi_R T_t f|\|_{L^q(\mu)}\\ &\le \int \|\sup_{t\in I_l}|T_t f_{\tau,l}|\|_{L^q(\mu)}\, d\tau\\ &\le CR^\beta \int \|f_{\tau,l}\|_2\, d\tau.\end{aligned}$$

Hence, (7) becomes bounded as

$$\begin{aligned}\|\sup_{0\le t\le R^k}|T_t f|\|_{L^q(\mu)} &\le CR^\beta \left[\sum_l \left(\int \|f_{\tau,l}\|_2\, d\tau\right)^q\right]^{1/q}\\ &\le CR^\beta \int \Big(\sum_l \|f_{\tau,l}\|_2^2\Big)^{1/2} d\tau\end{aligned}$$

where we used $q \ge 2$ and Minkowski's integral inequality. Thus, to prove (6) it suffices to prove that

$$\int_{|\tau|\le 10R^{-1}} \Big(\sum_l \|f_{\tau,l}\|_2^2\Big)^{1/2} d\tau \le C\|f\|_2. \tag{10}$$

For that, by Fubini's theorem, we first write

$$\sum_l \|f_{\tau,l}\|_2^2 = \int \Big(\sum_l |\widehat{f}_{\tau,l}(\xi)|^2\Big)\, d\xi.$$

In (9), we notice that

$$\widehat{\psi}_{R,l}(P(\xi)-P(\eta)+\tau) = Re^{-2\pi i\tau Rl} e^{2\pi i(P(\eta)-P(\xi))Rl}\, \widehat{\psi}(R(P(\xi)-P(\eta)+\tau)),$$

so the absolute value of (9) is

$$|\widehat{f}_{\tau,l}(\xi)| = R\left|\int e^{2\pi i(P(\eta)-P(\xi))Rl}\, a_{R,\tau}(\eta,\xi)\, \widehat{f}(\eta)\, d\eta\right|,$$

where the amplitude $a_{R,\tau}(\eta,\xi) := \widehat{\psi}(R(P(\xi)-P(\eta)+\tau))\, \widehat{\varphi}_R(\xi-\eta)$ is a smooth function and, as function of η, it is supported in $B(\xi,(c_0R)^{-1})$.

Let us fix ξ and τ. Observe that $\{\widehat{f}_{\tau,l}(\xi)\}_{l\in\mathbb{Z}}$ are not far from the Fourier coefficients of $a_{R,\tau}(\cdot,\xi)\widehat{f}$ at frequencies $\{\nabla P(\xi)Rl\}_{l\in\mathbb{Z}}$. In the same way as we can prove the inequality $\int|\widehat{F}|^2\mathbb{1}_E\, dx \le \|F\|_2^2$

using the crude estimate $\mathbb{1}_E \leq 1$, and the result is best possible, we can bound the square norm of $\{\widehat{f}_{\tau,l}(\xi)\}_{l\in\mathbb{Z}}$.

Assume that $\nabla P(\xi)$ points in the direction e_n, which can always be achieved after a rotation, if needed. Let $m = (m', m_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}$ and define the Fourier coefficients

$$b(m) := R \int e^{2\pi i R((\eta'-\xi')\cdot m' + (P(\eta)-P(\xi))m_n)} a_{R,\tau}(\eta,\xi)\widehat{f}(\eta)\, d\eta,$$

where $\xi = (\xi', \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Then,

$$\begin{aligned}\sum_{l\in\mathbb{Z}} |\widehat{f}_{\tau,l}(\xi)|^2 &\leq \sum_{m\in\mathbb{Z}^n} |b(m)|^2 \\ &= R^2 \int_{\mathbb{R}^{2n}} a_{R,\tau}(\eta,\xi)\, \widehat{f}(\eta)\, \overline{a_{R,\tau}(\omega,\xi)\, \widehat{f}(\omega)} \Big(\sum_m e^{2\pi i R[(\eta'-\omega')\cdot m' + (P(\eta)-P(\omega))m_n]} \Big)\, d\eta d\omega\end{aligned} \tag{11}$$

By the Poisson summation formula, the inner sum is

$$\sum_m e^{2\pi i R[(\eta'-\omega')\cdot m' + (P(\eta)-P(\omega))m_n]} = R^{-n} \sum_{m\in\mathbb{Z}^n} \delta_{m/R}(\eta' - \omega', P(\eta) - P(\omega)).$$

Since $a_{R,\tau}(\cdot,\xi)$ is supported in the ball $B(\xi, 1/(c_0R))$, we may work with $\eta, \omega \in B(\xi, 1/(c_0R))$. Let us check which Dirac deltas contribute to the integral (11). Suppose that $(\eta' - \omega', P(\eta) - P(\omega)) = m/R$ for some $m \in \mathbb{Z}^n$. Then, $|m'|/R = |\eta' - \omega'| \leq 2/(c_0R)$, which implies $m' = 0$ if we set $c_0 > 2$. Similarly, $|m_n|/R = |P(\eta) - P(\omega)| \leq 2\sup_{|\xi|\simeq 1}|\nabla P(\xi)|/(c_0R) \leq 1/(5R)$ implies $m_n = 0$. Hence, only the Dirac delta at $m = 0$ in (11) survives.

We have thus

$$\sum_{l\in\mathbb{Z}} |\widehat{f}_{\tau,l}(\xi)|^2 \leq R^{2-n} \int_{\mathbb{R}^{n+1}} a_{R,\tau}(\eta,\xi)\, \widehat{f}(\eta)\, \overline{a_{R,\tau}((\eta',\omega_n),\xi)\, \widehat{f}((\eta',\omega_n))}\, \delta_0(P(\eta) - P((\eta',\omega_n)))\, d\omega_n\, d\eta. \tag{12}$$

Call $G(\omega_n) = a_{R,\tau}(\eta,\xi)\, \widehat{f}(\eta)\overline{a_{R,\tau}((\eta',\omega_n),\xi)\, \widehat{f}((\eta',\omega_n))}$ so that the integral in ω_n is

$$\int_{\mathbb{R}} G(\omega_n)\, \delta_0(P(\eta) - P((\eta',\omega_n)))\, d\omega_n. \tag{13}$$

We want to change variables $r = P(\eta) - P((\eta',\omega_n))$, with $dr = -\partial_n P((\eta',\omega_n))\, d\omega_n$, where

$$\begin{aligned}-\partial_n P((\eta',\omega_n)) &= -\partial_n P(\xi) + (\partial_n P(\xi) - \partial_n P((\eta',\omega_n))) \\ &= -\partial_n P(\xi) + \mathcal{O}(1/(c_0R)).\end{aligned}$$

The last equality is due to the regularity of P and the support of $a_{R,\tau}(\cdot,\xi)$, since

$$\big|\partial_n P(\xi) - \partial_n P(\eta',\omega_n)\big| \leq c_2|\xi - (\eta',\omega_n)| \leq \frac{c_2}{c_0R}.$$

Now, recall that $\nabla P(\xi)$ points in the direction e_n, so $|\partial_n P(\xi)| = |\nabla P(\xi)|$. Since P is non-singular by hypothesis, we have $\inf_{|\xi|\simeq 1}|\nabla P(\xi)| \geq c_1 > 0$, and thus, choosing c_0 as large as needed, we get $|\partial_n P((\eta',\omega_n))| \geq c_1/2 > 0$. Consequently, the change of variables is bijective, so (13) turns into

$$\int_{\mathbb{R}} \frac{G(\omega_n(r))\, \delta_0(r)}{|\partial_n P((\eta',\omega_n(r)))|}\, dr = \frac{G(\omega_n(0))}{|\partial_n P((\eta',\omega_n(0)))|} \leq \frac{2}{c_1}\, G(\eta_n).$$

In the last inequality we used $\omega_n(0) = \eta_n$, because $0 = r = P(\eta) - P((\eta',\omega_n))$ and $r(\omega_n)$ is one-to-one.

Going back to (12),

$$\sum_l |\widehat{f}_{\tau,l}(\xi)|^2 \leq \frac{2}{c_1}\, R^{2-n} \int_{\mathbb{R}^n} |a_{R,\tau}(\eta,\xi)\widehat{f}(\eta)|^2\, d\eta \leq \frac{2}{c_1}\, \|\widehat{\psi}\|_\infty\, R^{2-n} \int_{\mathbb{R}^n} |\widehat{\varphi}_R(\xi-\eta)\widehat{f}(\eta)|^2\, d\eta.$$

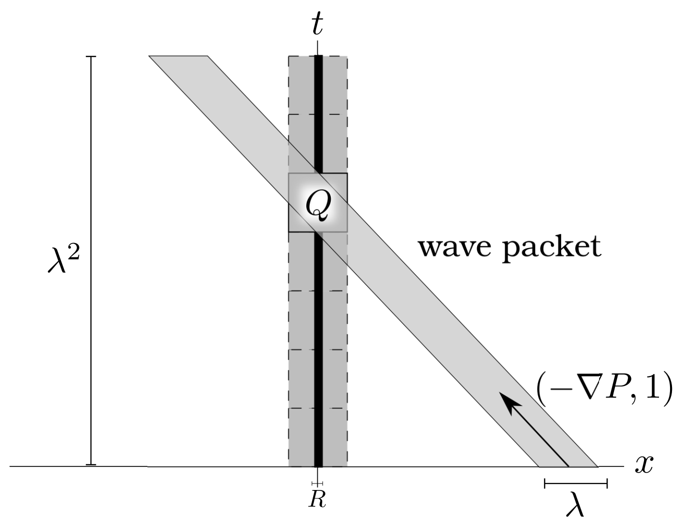


FIGURE 3. Lemma 2.1 can be proved using a wave packet decomposition. We find it easier to go through scales λ from $\lambda = R$ up to $\lambda = R^{k/2}$. We divide the cylinder $B_\lambda \times [0, \lambda^2]$ into cubes Q and assign to each cube all the wave packets passing through it, and then extract the corresponding contribution f_Q from the initial datum f. We apply the inductive hypothesis to each $\sup_{m\lambda \le t \le (m+1)\lambda} |T_t f_Q|$. All f_Q's are essentially orthogonal, so Bessel's inequality holds, *i.e.* something like (10).

where in the last inequality we removed the dependence on τ. Now integrate in ξ to get

$$\sum_l \|f_{\tau,l}\|_2^2 \le \frac{C_\psi}{c_1} R^{2-n} \int |\widehat{f}(\eta)|^2 \left(\int |\varphi_R(\xi)|^2 \, d\xi \right) d\eta = C_{\psi,\varphi} \frac{c_0^n}{c_1} R^2 \|f\|_2^2.$$

Finally, take the square root and integrate in τ, recalling that $|\tau| \le 10R^{-1}$. This proves (10), hence the lemma is proved. $\square$

A proof based on wave packets might be more intuitive, so we sketch in Figure 3 a proof of a slightly weaker result than Lemma 2.1.

Now, we perform a further reduction as in [16, p. 845-846].

Lemma 2.2. *Let $W^\alpha(B_R)$ be the collection of weights $w : B_R \subset \mathbb{R}^{n+1} \to \mathbb{R}$ with the following properties:*

(i) $w \ge 0$.
(ii) $\int_{B_r(x)} w \le Cr^\alpha$ holds for every $x \in \mathbb{R}^{n+1}$.
(iii) $w = \sum_Q w_Q \mathbb{1}_Q$, where Q are unit cubes that tile $\mathbb{R}^{n+1}$.
(iv) Every line $t \mapsto (x,t)$ intersects at most one cube in the support of w.

Suppose that for every $w \in W^\alpha(B_R)$ it holds that

$$\|T_t f\|_{L^2(w)} \le C_w R^\beta \|f\|_2, \tag{14}$$

where $\operatorname{supp} \widehat{f} \subset \{|\xi| \simeq 1\}$. *Then, for every $\mu \in M^\alpha(B_R)$ it also holds that*

$$\| \sup_{0 \le t \le R} |T_t f| \|_{L^2(\mu)} \le C_\mu R^\beta \|f\|_2. \tag{15}$$

Proof. Since the Fourier transform of $|T_t f|^2$ is supported in B_1, then we can find $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $|T_t f(x)|^2 = |T_s f|^2 * \psi(x,t)$. If we define $\psi_1(x,t) := \sup_{|(y,s)| \le \sqrt{n}} |\psi((x,t) + (y,s))|$, which has the same decay properties as ψ, then for every pair of points (x,t) and (X,T) separated by a distance less than $\sqrt{n}$ the inequality $|T_t f(x)|^2 \le |T_s f|^2 * \psi_1(X,T)$ holds true.

Let us tile $\mathbb{R}^{n+1}$ with unit cubes $Q = X \times T \subset \mathbb{R}^n \times \mathbb{R}$, whose center we denote also as (X,T). According to the previous paragraph, the value at (x,t) is controlled by the value at the center of the cube where it lies, that is,

$$|T_t f(x)|^2 \le \sum_Q |T_s f|^2 * \psi_1(X,T)\, \mathbb{1}_Q(x,t).$$

For every X, we can find some $T(X)$ such that

$$\sup_{0\le t\le R} |T_t f(x)|^2 \le \sum_X |T_s f|^2 * \psi_1(X,T(X))\, \mathbb{1}_X(x).$$

We integrate over μ to reach

$$\| \sup_{0\le t\le R} |T_t f| \|^2_{L^2(\mu)} \le \sum_X |T_s f|^2 * \psi_1(X,T(X))\, \mu(X). \tag{16}$$

Write now

$$\psi_1(x,t) \le \sum_Q b_Q\, \mathbb{1}_Q(x,t),$$

where $b_Q = \sup_{(x,t)\in Q} \psi_1(x,t)$ decay rapidly because ψ_1 decays faster than any power. Plug it into (16) and rearrange terms so that

$$\| \sup_{0\le t\le R} |T_t f(x)| \|^2_{L^2(\mu)} \le \sum_{Q'} b_{Q'} \int |T_t f(x)|^2 \Big(\sum_X \mathbb{1}_{Q'}(x-X, t-T(X))\, \mu(X) \Big)\, dxdt.$$

Define the weight

$$w(x,t) := \sum_X \mathbb{1}_{X\times T(X)}(x,t)\, \mu(X),$$

which satisfies all the properties listed in the statement of the lemma. With it, if $Q' = X' \times T'$, we can write

$$\begin{aligned} \| \sup_{0\le t\le R} |T_t f(x)| \|^2_{L^2(\mu)} &\le \sum_{Q'} b_{Q'} \int |T_t f(x)|^2\, w(x-X', t-T')\, dxdt \\ &= \sum_{Q'} b_{Q'} \int |T_{t+T'} f(x+X')|^2\, w(x,t)\, dxdt. \end{aligned}$$

Observe that

$$T_{t+T'} f(x+X') = T_t(f_{X',T'})(x) \qquad \text{such that} \qquad \widehat{f_{X',T'}}(\xi) = \widehat{f}(\xi)\, e^{i\,(X'\xi + iT'P(\xi))}.$$

Thus, using the hypothesis (14), we bound

$$\| \sup_{0\le t\le R} |T_t f(x)| \|^2_{L^2(\mu)} \le C_w^2\, R^{2\beta} \sum_{Q'} b_{Q'}\, \|f_{X',T'}\|_2^2 = C_w^2\, R^{2\beta}\, \|f\|_2^2 \sum_{Q'} b_{Q'}.$$

The fact that b_Q decay rapidly implies (15). ☐

In the next lemma we summarize the reductions until now.

Lemma 2.3. *Suppose that for every $w \in W^\alpha(B_R)$ as defined in Lemma 2.2, and every $R \gg 1$ it holds that*

$$\|T_t f\|_{L^2(w)} \le C R^{s(\alpha)-(n-\alpha)/2} \|f\|_2,$$

where $\operatorname{supp} \widehat{f} \subset \{|\xi| \simeq 1\}$. Then, for every $\mu \in M^\alpha(B_1)$ the solution $T_t f$ converges μ-a.e. to $f \in H^s(\mathbb{R}^n)$ if $s > s(\alpha)$.

With all these reduction we can prove the following theorem, one of the main results of this section.

Theorem 1.1. *If $s > (n-\alpha+1)/2$ and $\mu \in M^\alpha(B_1)$, then $T_t f$ converges to f μ-a.e. for every $f \in H^s(\mathbb{R}^n)$.*

Proof. By Lemma 2.3, it suffices to prove that

$$\|T_t f\|_{L^2(w)} \leq R^{1/2}\, \|w\|_\infty^{1/2}\, \|f\|_2, \tag{17}$$

where $w \in W^\alpha(B_R)$. Using Plancherel's theorem and the fact that w is bounded, for every time t we have

$$\int |T_t f(x)|^2\, w(x,t)\, dx \leq \|w\|_\infty \|f\|_2^2.$$

Integrating in $t \in [-R, R]$ we get the desired inequality (17). $\square$

In this generality, Theorem 1.1 is best possible for $\alpha > 1$. For example, for the transport equation (17) cannot be improved. What is more, some dispersive symbols like the saddle behave as a non-dispersive symbol with especially crafted initial data, as we see in Theorem 1.5.

Recall that for $s < (n-\alpha)/2$ convergence does not hold. In general, for a fixed symbol P, it is a very hard problem to close the gap between convergence and divergence. However, if the symbol is dispersive, then we can close it when α is small, as we show now.

Theorem 1.2. *Suppose there exists $\beta > 0$ such that $\|T_t\varphi\|_\infty \leq C_\varphi |t|^{-\beta}$ for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with Fourier support in $\{|\xi| \simeq 1\}$. Then, for $\alpha < \beta$ and $s > (n-\alpha)/2$, $T_t f$ converges to f μ-a.e. for every $f \in H^s(\mathbb{R}^n)$ and for every $\mu \in M^\alpha(B_1)$.*

Proof. We use Lemma 2.3 to prove this. It suffices to work with data f Fourier supported in the annulus $\{|\xi| \simeq 1\}$, so let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a cut-off function such that $\varphi = 1$ in that annulus and write

$$T_t f(x) = \int \varphi(\xi)\, \widehat{f}(\xi)\, e^{2\pi i (x\cdot\xi + tP(\xi))}\, d\xi.$$

We will first prove that

$$\|T_t f\|_{L^2(w)} \leq C\, \big\|\big|(dS)^\vee\big| * w\big\|_\infty^{1/2}\, \|f\|_2, \tag{18}$$

and then we will check that $\||(dS)^\vee| * w\|_\infty^{1/2} \leq C$, which by Lemma 2.3 implies the result.

By Plancherel's theorem, defining the operator E such that $Ef = T_t(f^\vee)$, it is enough to prove that

$$\|Ef\|_{L^2(w)} \leq C\, \big\|\big|(dS)^\vee\big| * w\big\|_\infty^{1/2}\, \|f\|_2. \tag{19}$$

By duality,

$$\|Ef\|_{L^2(w)}^2 = \sup_{\|gw^{1/2}\|_2 = 1} \langle Efw^{1/2}, gw^{1/2}\rangle \leq \|f\|_2^2 \sup_{\|g\|_{L^2(w)} = 1} \|E^*(gw)\|_2^2$$

where the adjoint operator E^* is essentially the restriction operator attached to S, that is,

$$E^*F(\xi) = \varphi(\xi) \int F(x,t)\, e^{-2\pi i (x\cdot\xi + tP(\xi))}\, dxdt.$$

Rearranging the integrals, we get

$$\|E^*(gw)\|_2^2 = \langle E^*(gw), E^*(gw)\rangle = \int gw\, \overline{(gw * (dS)^\vee)}\, dxdt \leq \|g\|_{L^2(w)}\, \|gw * (dS)^\vee\|_{L^2(w)},$$

where $(dS)^\vee(x,t) = \int \varphi(\xi)^2\, e^{i(x\xi + tP(\xi))}\, d\xi$. Thus, to prove (19) it is enough to prove

$$\|gw * (dS)^\vee\|_{L^2(w)} \leq C\, \big\|\big|(dS)^\vee\big| * w\big\|_\infty\, \|g\|_{L^2(w)}. \tag{20}$$

We prove (20) by interpolation between $L^\infty(w) \to L^\infty(w)$ and $L^1(w) \to L^1(w)$. Indeed, the L^∞ bound follows from

$$\begin{aligned} |gw * (dS)^\vee(x,t)| &\le \int \left|(gw)(y,s)(dS)^\vee(x-y,t-s)\right| \, dyds \\ &= \int \left|(g\mathbb{1}_{\operatorname{supp} w})(y,s)\, w(y,s)(dS)^\vee(x-y,t-s)\right| \, dyds \\ &\le \|g\|_{L^\infty(w)} \left(w * \left|(dS)^\vee\right|\right)(x,t), \end{aligned}$$

while the L^1 bound holds because

$$\begin{aligned} \int |(gw) * (dS)^\vee| \, w \, dxdt &\le \int |gw(y,s)| \int |w(x,t)(dS)^\vee(x-y,t-s)| \, dxdt \, dyds \\ &\le \|g\|_{L^1(w)} \, \|\left|(dS)^\vee\right| * w\|_\infty. \end{aligned}$$

This implies (20) and so (18) is proved.

Let us bound $\||(dS)^\vee| * w\|_\infty$ now. By the non-stationary phase principle,

$$|(dS)^\vee(x,t)| \le C \sup_{2\le j\le N} \|D^j P\|_\infty \, |x|^{-N}, \quad \text{for every } N \ge 1 \text{ and } |x| \ge 2c_0|t|,$$

where $c_0 := \sup_{|\xi|\simeq 1} |\nabla P(\xi)|$ and C depends on c_0 and φ, but is otherwise independent of time. On the other hand, by hypothesis we have $|(dS)^\vee(x,t)| \le C|t|^{-\beta}$. Combining the two bounds, we obtain

$$\left|(dS)^\vee(x,t)\right| \le \frac{C}{|(x,t)|^\beta}, \qquad \text{for any } (x,t),$$

so by decomposing the space in dyadic annuli we deduce that

$$|(dS)^\vee(x,t)| \le C \sum_{\lambda\ge 1,\, \lambda \text{ dyadic}} \lambda^{-\beta} \, \mathbb{1}_{B_\lambda}(x,t).$$

Hence,

$$\begin{aligned} \int \left|(dS)^\vee(x-y,t-s)\right| \, w(y,s)\, dyds &\le C \sum_{\lambda\ge 1} \lambda^{-\beta} \int_{B_\lambda(x,t)} w \\ &\lesssim \sum_{1\le\lambda\le R} \lambda^{-\beta+\alpha} + R^\alpha \sum_{R\le\lambda} \lambda^{-\beta} \\ &\le \sum_{\lambda\ge 1} \lambda^{-\beta+\alpha}, \end{aligned}$$

which is bounded if $\alpha < \beta$. In that case, $\||(dS)^\vee| * w\|_\infty \lesssim 1$, which we insert into (18) and apply Lemma 2.3 to conclude the proof. ☐

3. Building the counterexample

We begin the proof of Theorem 1.4. Recall that we consider symbols of the form $P(\xi) = \xi_1^k + W(\xi')$, where $W \in \mathbb{Q}[X_2, \cdots, X_n]$ is a homogeneous polynomial of degree $k \ge 2$. It suffices to work with $W \in \mathbb{Z}[X_2, \cdots, X_n]$; indeed, given that there exists $a \in \mathbb{Z}$ such that $aW \in \mathbb{Z}[X_2, \cdots, X_n]$, we study $T_{at}f(x)$ when $t \to 0$ instead of $T_t f(x)$.

3.1. **A preliminary datum.** Let $0 < c \ll 1$ and choose two positive functions $\phi_1 \in \mathcal{S}(\mathbb{R})$ and $\phi_2 \in \mathcal{S}(\mathbb{R}^{n-1})$ such that $\operatorname{supp} \widehat{\phi}_i \subset B(0,c)$. Let $\psi \in \mathcal{S}(\mathbb{R}^{n-1})$ be another positive function with $\operatorname{supp} \psi_i \subset B(0,1)$. For $R > 1$ and $D > 1$, let us define

$$f_R(x) = \phi_1(R^{1/2}x_1)\, e^{2\pi i R x_1}\, \phi_2(x') \sum_{m' \in \mathbb{Z}^{n-1}} \psi\left(\frac{Dm'}{R}\right) e^{2\pi i D m' \cdot x'} \tag{21}$$

where $x = (x_1, \ldots, x_n) = (x_1, x') \in B(0,1)$. Defining

$$g(x_1) = \phi_1(R^{1/2}x_1)\, e^{2\pi i R x_1} \qquad \text{and} \qquad h(x') = \phi_2(x') \sum_{m' \in \mathbb{Z}^{n-1}} \psi\left(\frac{Dm'}{R}\right) e^{2\pi i D m' \cdot x'},$$

we may write $f_R(x) = g(x_1)\, h(x')$. In particular, $\widehat{f_R}(\xi) = \widehat{g}(\xi_1)\, \widehat{h}(\xi')$, where

$$\widehat{g}(\xi_1) = R^{-1/2}\, \widehat{\phi}\big(R^{-1/2}(\xi_1 - R)\big) \qquad \text{and} \qquad \widehat{h}(\xi') = \sum_{m' \in \mathbb{Z}^{n-1}} \psi\left(\frac{Dm'}{R}\right) \widehat{\phi}_2(\xi' - Dm').$$

Thus, the $L^2(\mathbb{R}^n)$ norm of f_R is

$$\|f_R\|_2 \simeq R^{-1/4} \left(\frac{R}{D}\right)^{(n-1)/2}. \tag{22}$$

Let us analyze the evolution of f_R. In the x_1 variable, the evolution is

$$\begin{aligned} T_t g(x_1) &= R^{-1/2} \int \widehat{\phi}(R^{-1/2}\, \xi_1)\, e(x_1\,(\xi_1 + R) + t(\xi_1 + R)^k)\, d\xi_1 \\ &= e^{2\pi i (x_1 R + t R^k)} \int \widehat{\phi}(\xi_1)\, e\Big(R^{1/2}\, \xi_1 (x_1 + tkR^{k-1}) + t \sum_{j=2}^{k} \binom{k}{j} \xi_1^j\, R^{k-j/2}\Big)\, d\xi_1. \end{aligned} \tag{23}$$

Observe that $\left|\sum_{j=2}^{k} \binom{k}{j} \xi_1^j\, R^{k-j/2}\right| \le R^{k-1} \sum_{j=2}^{k} \binom{k}{j}$, so asking $|t| < 1/R^{k-1}$ we essentially have

$$|T_t g(x_1)| \simeq \left|\phi_1\left(R^{1/2}\big(x_1 + kR^{k-1}t\big)\right)\right|. \tag{24}$$

In the remaining variables x', the evolution is

$$\begin{aligned} T_t h(x') &= \sum_{m' \in \mathbb{Z}^{n-1}} \psi\left(\frac{Dm'}{R}\right) \int \widehat{\phi}_2(\xi')\, e\big(x' \cdot (\xi' + Dm') + t\, W(\xi' + Dm')\big)\, d\xi' \\ &= \sum_{m' \in \mathbb{Z}^{n-1}} e^{2\pi i (x' \cdot Dm' + tW(Dm'))} \\ &\qquad \psi\left(\frac{Dm'}{R}\right) \int \widehat{\phi}_2(\xi')\, e\left(x' \cdot \xi' + t(W(Dm' + \xi') - W(Dm'))\right)\, d\xi' \end{aligned} \tag{25}$$

Let us evaluate the solution at the points

$$t = \frac{p_1}{D^k\, q} \qquad \text{and} \qquad x' = \frac{p'}{D\, q} + \epsilon, \tag{26}$$

where $p = (p_1, p') \in \mathbb{Z}^n$ and $q \in \mathbb{N}$ with $q \simeq Q$ for some $Q \geq 1$ to be chosen later, and $|\epsilon| \lesssim R^{-1}$. Thus,

$$\begin{aligned} T_t h(x') = &\sum_{m' \in \mathbb{Z}^{n-1}} e\left(\frac{p' \cdot m' + p_1 W(m')}{q}\right) \\ &\qquad \left[e^{2\pi i\, \epsilon \cdot Dm'} \psi\left(\frac{Dm'}{R}\right) \int \widehat{\phi}_2(\xi')\, e\left(x' \cdot \xi' + t(W(Dm' + \xi') - W(Dm'))\right) d\xi' \right] \quad (27) \\ = &\sum_{m' \in \mathbb{Z}^{n-1}} e\left(\frac{p' \cdot m' + p_1 W(m')}{q}\right) \zeta(m'), \end{aligned}$$

where we define

$$\zeta(m') = e^{2\pi i\, \epsilon \cdot Dm'} \psi\left(\frac{Dm'}{R}\right) \int \widehat{\phi}_2(\xi')\, e\left(x' \cdot \xi' + t(W(Dm' + \xi') - W(Dm'))\right) d\xi'. \qquad (28)$$

Heuristically, if we assume that $R/D \gg Q$, the function ζ is roughly constant at scale q, so we should be able to sum in blocks modulo q. In that case, due to the homogeneity of the polynomial $W(m')$ and since the support of ζ is contained in $|m'| \leq R/D$, the solution would be

$$\left|T_t h(x')\right| \simeq \left(\frac{R}{DQ}\right)^{n-1} \left| \sum_{r' \in \mathbb{F}_q^{n-1}} e\left(\frac{p' \cdot r' + p_1 W(r')}{q}\right) \right|. \qquad (29)$$

At this point, we have two goals:

- To make (29) rigorous, and
- To estimate the exponential sum in (29).

As we shall see, estimating the exponential sum is necessary to prove (29), so let us begin by that.

First, notice that the exponential sum is the discrete inverse Fourier transform of the function $S(r_1, r') = \delta_0(W(r') - r_1)$ in the variables $(r_1, r') \in \mathbb{F}_q^n$, that is,

$$S(r_1, r') = \delta_0(W(r') - r_1) \qquad \Longrightarrow \qquad \check{S}(p) = \sum_{r' \in \mathbb{F}_q^{n-1}} e\left(\frac{p' \cdot r' + p_1 W(r')}{q}\right). \qquad (30)$$

Notice also that $\check{S}(p)$ is the extension operator attached to the surface $\{(W(r'), r') \in \mathbb{F}_q^n\}$.

Estimating this kind of sums is difficult in general. In [1], An, Chu and Pierce applied Deligne's Theorem 8.4 from [12], a deep result in arithmetic geometry, which we write here. The reader is also referred to Theorem 11.43 in [19].

Theorem 3.1 (Deligne's theorem adapted)**.** *Let $f \in \mathbb{Z}[X_1, \ldots, X_d]$ be a nonzero polynomial of degree k with homogeneous part f_k. Suppose that*

(i) q is a prime number and $q \nmid k$.

(ii) $\nabla f_k(x) \neq 0$ for every $x \in \overline{\mathbb{F}}_q^{\,d} \setminus \{0\}$.

Then,

$$\left| \sum_{x \in \mathbb{F}_q^d} e(f(x)/q) \right| \leq (k-1)^d q^{d/2}. \qquad (31)$$

We first remark that we need (31) to hold for all but finitely many primes q. In this setting, if a polynomial $f \in \mathbb{Z}[X_1, \ldots, X_d]$ satisfies the hypotheses of Deligne's theorem for all but finitely many primes, then we say that f satisfies the Weil bound.

Our second remark is that if f is homogeneous of degree ≥ 2 and satisfies the Weil bound, then the polynomial $af(x) + l(x)$ with any integer a which is coprime with q and any linear form l also satisfies the Weil bound. Thus, for our sum in (30) it is enough to check that W satisfies the Weil bound.

Let us comment on condition (ii) now, about which the reader can find a discussion in [34]. It is equivalent to the statement that the variety (scheme) defined by f_k in $\mathbb{P}^{d-1}(\mathbb{F}_q)$ is smooth. However, we want to find conditions that make a polynomial satisfy the Weil bound that are easier to verify than condition (ii). We do that in the following corollary.

Corollary 3.2. *Let $f \in \mathbb{Z}[X_1, \dots, X_d]$ be a homogeneous polynomial. If $\nabla f(x) \neq 0$ for every $x \in \mathbb{C}^d \setminus \{0\}$, then f satisfies the Weil bound, that is, (31) holds for all but finitely many primes.*

To prove it, we need Hilbert's Nullstellensatz, which we recall here in the version written in [43, Theorem 14, Ch. VII.3].

Theorem 3.3 (Hilbert's Nullstellensatz)**.** *Let k be a field. If $F, F_1, \dots, F_m$ are polynomials in $k[X_1, \dots, X_d]$ and if F vanishes at every common zero of $F_1, \dots, F_m$ in an algebraically closed extension K of k, then there exists an exponent $\rho \in \mathbb{N}$ and polynomials $A_1, \dots, A_m \in k[X_1, \dots, X_d]$ such that*

$$F^\rho = A_1 F_1 + \dots + A_m F_m.$$

Proof of Corollary 3.2. We must prove that $\nabla f(x) \neq 0$ for every $x \in \overline{\mathbb{F}_q}^d \setminus \{0\}$ and all but finitely many primes q.

By Hilbert's Nullstellensatz with $k = \mathbb{Q}$ and $K = \mathbb{C}$, we can find a natural number N and polynomials $A_{ij} \in \mathbb{Q}[X_1, \dots, X_d]$, for $i, j = 1, \dots, d$, such that

$$X_i^N = A_{i1}\, \partial_1 f + \dots + A_{id}\, \partial_d f, \qquad \text{for all } i = 1, \dots, d.$$

We multiply the polynomials above by some $M \in \mathbb{Z}$ such that $MA_{ij} \in \mathbb{Z}[X_1, \dots, X_d]$ for every i, j, and hence

$$M\, X_i^N = (MA_{i1})\, \partial_1 f + \dots + (MA_{id})\, \partial_d f, \qquad \text{for all } i = 1, \dots, d. \tag{32}$$

These are equalities among polynomials with integer coefficients. Thus, for every prime $q \nmid M$ (in particular, for $q > M$) we can reduce modulo q, so that (32) holds in $\mathbb{F}_q$. This implies that if $\nabla f(x) = 0$ for some $x \in \overline{\mathbb{F}_q}^d$, then necessarily $x = 0$. The proof is thus complete. $\square$

Thus, Corollary 3.2 implies that every homogeneous polynomial $W \in \mathbb{Z}[X_2, \dots, X_n]$ such that $\nabla W(x) \neq 0$ for every $x \in \mathbb{C}^{n-1} \setminus \{0\}$ satisfies the Weil bound. In that case, we bound (30) by

$$\left|\check{S}(p)\right| = \left| \sum_{r' \in \mathbb{F}_q^{n-1}} e\left(\frac{p' \cdot r' + p_1 W(r')}{q} \right) \right| \leq (k-1)^{n-1}\, q^{(n-1)/2} \tag{33}$$

for every $p_1 \not\equiv 0 \pmod{q}$ and for all but finitely many primes q.

Once we have estimated the exponential sum, let us justify (29). We do that in an abstract setting, like we did with the exponential sum.

Lemma 3.4. *Let $f \in \mathbb{Z}[X_1, \dots, X_d]$ be a polynomial of degree ≥ 2 that satisfies the Weil bound. Let also $\zeta \in C_0^\infty$ and define the discrete Laplacian $\widetilde{\Delta}$ by*

$$\widetilde{\Delta}\zeta(m) = \sum_{j=1}^{d} \big(\zeta(m + e_j) + \zeta(m - e_j) - 2\zeta(m) \big).$$

Assume that, for some $L > 0$, ζ is supported in $B(0, L)$ and $\|\widetilde{\Delta}^N \zeta\|_\infty \lesssim_N L^{-2N}$ for every $N \in \mathbb{N}$. Then,

$$\sum_{m \in \mathbb{Z}^d} \zeta(m) e(f(m)/q) = \left(\frac{1}{q^d} \sum_{m \in \mathbb{Z}^d} \zeta(m) \right) \sum_{l \in \mathbb{F}_q^d} e(f(l)/q) + \mathcal{O}_N \left(q^{d/2} \left(\frac{L}{q} \right)^{d-2N} \right) \tag{34}$$

for any integer $N > d/2$.

Proof. We rearrange the sum into blocks modulo q so that

$$\sum_{m\in\mathbb{Z}^d} \zeta(m) e(f(m)/q) = \sum_{r\in\mathbb{F}_q^d} \left(\sum_{m\equiv r \pmod q} \zeta(m) \right) e(f(r)/q) = \sum_{r\in\mathbb{F}_q^d} Z(r)\, e(f(r)/q), \tag{35}$$

where we define $Z(r) = \sum_{m\equiv r \pmod q} \zeta(m)$ for every $r \in \mathbb{F}_q^d$. Decompose Z into frequencies

$$\widehat{Z}(\omega) = \frac{1}{q^d} \sum_{r\in\mathbb{F}_q^d} Z(r) e^{-2\pi i \omega\cdot r/q} \qquad \text{such that} \qquad Z(r) = \sum_{\omega\in\mathbb{F}_q^d} \widehat{Z}(\omega) e^{2\pi i \omega\cdot r/q},$$

and replace them into (35) so that

$$\sum_{m\in\mathbb{Z}^d} \zeta(m) e(f(m)/q) = \widehat{Z}(0) \sum_{m\in\mathbb{F}_q^d} e(f(m)/q) + \sum_{\omega\in\mathbb{F}_q^d\setminus\{0\}} \widehat{Z}(\omega) \left(\sum_{r\in\mathbb{F}_q^d} e\left(\frac{f(r)+\omega\cdot r}{q} \right) \right).$$

Since $\widehat{Z}(0) = q^{-d} \sum_{m\in\mathbb{Z}^d} \zeta(m)$, to prove (34) we need to control the error term

$$E = \sum_{\omega\in\mathbb{F}_q^d\setminus\{0\}} \widehat{Z}(\omega) \left(\sum_{r\in\mathbb{F}_q^d} e\left(\frac{f(r)+\omega\cdot r}{q} \right) \right).$$

Observe that

$$|E| \le \|\widehat{Z}\|_{\ell^1(\mathbb{F}_q^d\setminus\{0\})} \sup_{\omega\in\mathbb{F}_q^d} \left| \sum_{r\in\mathbb{F}_q^d} e\left(\frac{f(r)+\omega\cdot r}{q} \right) \right|$$

From the Weil Bound in Theorem 3.1 we know that

$$\Big| \sum_{r\in\mathbb{F}_q^d} e\left(\frac{f(r)+\omega\cdot r}{q} \right) \Big| \lesssim q^{d/2}.$$

To bound $\widehat{Z}(\omega)$, we sum by parts using the discrete Laplacian $\widetilde{\Delta}$. First, observe that for a fixed ω,

$$\begin{aligned} -\widetilde{\Delta}\left(e^{-2\pi i\omega\cdot r/q}\right) &= \sum_{j=1}^{d} \left(2e^{-2\pi i\,\omega\cdot r/q} - e^{-2\pi i\,\omega\cdot(r+e_j)/q} - e^{-2\pi i\,\omega\cdot(r-e_j)/q} \right) \\ &= e^{-2\pi i\omega\cdot r/q} \sum_{j=1}^{d} \sin^2(\pi\omega_j/q) = e^{-2\pi i\omega\cdot r/q} A(\omega), \end{aligned}$$

where $\{e_j\}_j$ is the canonical basis and $A(\omega) = 4\sum_{j=1}^d \sin^2(\pi\omega_j/q)$. Thus,

$$\widehat{Z}(\omega) = -\frac{1}{q^d\, A(\omega)} \sum_{r\in\mathbb{F}_q^d} Z(r)\, \widetilde{\Delta}\left(e^{-2\pi i\omega\cdot r/q}\right) = -\frac{1}{q^d\, A(\omega)} \sum_{r\in\mathbb{F}_q^d} \widetilde{\Delta}\left(Z(r)\right) e^{-2\pi i\omega\cdot r/q}.$$

Iterating, we get

$$\widehat{Z}(\omega) = \frac{(-1)^N}{q^d\, (A(\omega))^N} \sum_{r\in\mathbb{F}_q^d} \widetilde{\Delta}^N \left(Z(r)\right) e^{-2\pi i\omega\cdot r/q}, \qquad \forall N\in\mathbb{N}.$$

Since $\sin^2(\pi x) \geq c\|x\|^2$, where $\|x\|$ is the distance of x to the closest integer, then $A(\omega) \gtrsim \sum_{j=1}^d \|\omega_j/q\|^2$, so

$$\left|\widehat{Z}(\omega)\right| \lesssim \frac{1}{q^d}\left(\sum_{j=1}^d \|\omega_j/q\|^2\right)^{-N}\left|\sum_{r\in\mathbb{F}_q^d} \widetilde{\Delta}^N\left(Z(r)\right)e^{-2\pi i\omega\cdot r/q}\right|$$
$$\leq \left(\sum_{j=1}^d \|\omega_j/q\|^2\right)^{-N}\|\widetilde{\Delta}^N Z\|_\infty, \qquad \forall N\in\mathbb{N}.$$

As long as $2N > d$, we can write

$$\sum_{\omega\in\mathbb{F}_q^d\setminus\{0\}}\left(\sum_{j=1}^d \|\omega_j/q\|^2\right)^{-N} \simeq \sum_{\omega\in\{1,\dots,\frac{q-1}{2}\}^d}\frac{q^{2N}}{|\omega|^{2N}} \simeq_N q^{2N},$$

so

$$\|\widehat{Z}\|_{\ell^1(\mathbb{F}_q^d\setminus\{0\})} \lesssim q^{2N}\,\|\widetilde{\Delta}^N Z\|_\infty, \qquad \forall N\in\mathbb{N},$$

and thus,

$$|E| \lesssim q^{2N+d/2}\|\widetilde{\Delta}^N Z\|_\infty, \qquad \forall N\in\mathbb{N}.$$

Now, from the definition of Z and the hypotheses for ζ we see that

$$|\widetilde{\Delta}^N Z(r)| = \Big|\sum_{l\in\mathbb{Z}^d}\widetilde{\Delta}^N\zeta(lq+r)\Big| \lesssim \left(\frac{L}{q}\right)^d\|\widetilde{\Delta}^N\zeta\|_\infty \lesssim_N \frac{L^{d-2N}}{q^d}, \qquad \forall r\in\mathbb{F}_q^d,$$

so

$$|E| \lesssim_N q^{d/2}\left(\frac{L}{q}\right)^{d-2N}.$$

□

We can now rigorously estimate $T_t h$ as it appeared in (27). For that, we use Lemma 3.4 with $d = n-1$ and $L = R/D$. We only need to bound $\widetilde{\Delta}^N\zeta$ as indicated in the hypotheses for the function ζ defined in (28).

Lemma 3.5. *Let ζ be defined as in (28), and let $L = R/D$. Then $\|\widetilde{\Delta}^N\zeta\|_\infty \lesssim_N L^{-2N}$ for every $N\in\mathbb{N}$.*

Proof. Since $\|\widetilde{\Delta}^N\zeta\|_\infty \lesssim \sup_{|\alpha|=2N}\|\partial^\alpha\zeta\|_\infty$, where $\alpha = (\alpha_1,\dots,\alpha_{n-1})$ is a multi-index, it suffices to bound

$$\partial^\alpha\zeta(m') = \int\widehat{\phi}_2(\xi')\,e^{2\pi i x'\cdot\xi'}\,\partial^\alpha_{m'}\left[\psi\left(\frac{m'}{L}\right)e\left(\epsilon\cdot m'\frac{R}{L}+t\left(W\left(\frac{R}{L}m'+\xi'\right)-W\left(\frac{R}{L}m'\right)\right)\right)\right]d\xi'$$
$$= \int\widehat{\phi}_2(\xi')\,e^{2\pi i x'\cdot\xi'}\,\partial^\alpha_{m'}\left[\psi\left(\frac{m'}{L}\right)e\left(\delta\cdot\frac{m'}{L}+R\tau\left(W\left(\frac{m'}{L}+\frac{\xi'}{R}\right)-W\left(\frac{m'}{L}\right)\right)\right)\right]d\xi',$$

where $\delta = R\epsilon$ and $\tau = R^{k-1}t$, so that $|\delta|\leq 1$ and $|\tau|\leq 1$. If we define the phase function

$$F(z') = \epsilon\cdot z' + R\tau\left(W(z'+\xi'/R)-W(z')\right),$$

since for a function G we have $\partial^\alpha_{m'}\left(G(m'/L)\right) = L^{-2N}\,\partial^\alpha G(m'/L)$, it suffices to prove that

$$\left|\partial^\alpha_{z'}\Big[\psi(z')\,e(F(z'))\Big]\right| \lesssim 1 \qquad \text{uniformly for } |z'|\leq 1.$$

Since $|\partial^\alpha F(z')| \lesssim 1$ uniformly in z', ϵ, τ, R, and ξ', then the result follows from the Leibniz rule. □

Thus, we apply Lemma 3.4 with the first integer N that satisfies $N > (n-1)/2$. Due to the smallness of all phases in the definition of ζ, we get $\sum_{m'\in\mathbb{Z}^{n-1}} \zeta(m') \simeq (R/D)^{n-1}$, so we obtain

$$\left|T_t h(x')\right| = \left(\frac{R}{DQ}\right)^{n-1} |\breve{S}(p)| + \mathcal{O}\left(Q^{(n-1)/2}\left(\frac{R}{DQ}\right)^{-1}\right). \tag{36}$$

Now, even if in (33) we estimated $\breve{S}(p)$ from above, we also need an estimate from below. With the aid of Deligne's theorem, An, Chu and Pierce showed in [1, Proposition 2.2] that $|\breve{S}(p)| \gtrsim q^{(n-1)/2}$ for most $p \in \mathbb{F}_q^n$. For completeness, we repeat here their arguments.

Lemma 3.6. *If W satisfies the Weil Bound, then, for $q \gg 1$,*

$$|\breve{S}(p)| = \left| \sum_{r'\in\mathbb{F}_q^{n-1}} e\left(\frac{p'\cdot r' + p_1 W(r')}{q}\right)\right| \geq \frac{1}{10}\, q^{(n-1)/2}$$

for every $p \in G(q) \subset \mathbb{F}_q^n$, where $|G(q)| \geq C_{k,n} q^n$.

Proof. Following (30), Plancherel's theorem gives

$$\sum_{p\in\mathbb{F}_q^n} |\breve{S}(p)|^2 = q^n \sum_{r\in\mathbb{F}_q^n} \delta(W(r') - r_1) = q^{2n-1}.$$

For $c_1 > 0$, define $G(q) = \{p \in \mathbb{F}_q^n \mid |\breve{S}(p)| \geq c_1 q^{(n-1)/2}\}$. Since the Weil bound implies $|\breve{S}(p)| \leq (k-1)^{n-1} q^{(n-1)/2}$, we have

$$q^{2n-1} = \sum_{p\in\mathbb{F}_q^n} |\breve{S}(p)|^2 \leq q^{2n-2} + |G(q)\setminus\{0\}|(k-1)^{2(n-1)} q^{n-1} + c_1^2\, q^{2n-1}.$$

By choosing c_1 small enough we obtain $|G(q)| \gtrsim q^n$. $\square$

Applying Lemma 3.6 in (36), at every $p \in G(q)$ one gets

$$|T_t h(x')| \gtrsim \left(\frac{R}{DQ}\right)^{n-1} Q^{(n-1)/2} + \mathcal{O}\left(Q^{(n-1)/2}\left(\frac{R}{DQ}\right)^{-1}\right) \simeq \left(\frac{R}{DQ^{1/2}}\right)^{n-1}, \tag{37}$$

where we require $R/D \gg Q$ for the last operation. We remark that the case $q = Q = 1$, which is relevant in some parts of the following sections, is not covered by the arguments above. However, (37) still holds by direct computation.

We join estimates (24) and (37) to obtain

$$|T_t f_R(x)| \gtrsim \left|\phi_1(R^{1/2}(x_1 + kR^{k-1}t))\right| \left(\frac{R}{DQ^{1/2}}\right)^{n-1},$$

for every $x = (x_1, x') \in [-1,1]^n$ and $t < 1/R^{k-1}$ like in (26). That means that if we also ask for $|x_1 + kR^{k-1}t| < R^{-1/2}$, we get

$$|T_t f_R(x)| \gtrsim \left(\frac{R}{DQ^{1/2}}\right)^{n-1}. \tag{38}$$

We gather all such requirements for x in the following definition.

Definition 3.7 (Admissible Slabs)**.** Let $k \in \mathbb{N}$, $R > 1$, $D > 1$ and $Q > 1$, and let $G(q) \subset \mathbb{F}_q^n$ be the set given by Lemma 3.6. The slab

$$E_{p,q,R} = B_1\left(k\,\frac{R^{k-1}}{D^k}\frac{p_1}{q}, \frac{1}{R^{1/2}}\right) \times B_{n-1}\left(\frac{1}{D}\,\frac{p'}{q}, \frac{1}{R}\right) \tag{39}$$

is admissible whenever $Q/2 \leq q \leq Q$ and $p = (p_1, p') = (p_1, \ldots, p_n) \in G(q)$. Here, when we write $p \in G(q)$ we mean $p \pmod q \in G(q)$.

The definition of these slabs allows us to synthesize the discussion so far in the next proposition, which combines (22) and (38). The conclusion is similar to that in [1, Proposition 5.1].

Proposition 3.8. *Let $k \in \mathbb{N}$, $R > 1$, $D > 1$ and $Q > 1$, and let $W \in \mathbb{Q}[X_2, \cdots, X_n]$ be such that $\nabla W(x) \neq 0$ whenever $x \in \mathbb{C}^{n-1} \setminus \{0\}$. Let also f_R be the initial datum (21). If $t = p_1/(D^k q)$, $q \simeq Q$ and $E_{p,q,R}$ is an admissible slab, then*

$$\frac{|T_t f_R(x)|}{\|f_R\|_2} \gtrsim R^{1/4} \left(\frac{R}{DQ}\right)^{\frac{n-1}{2}} \qquad \text{for all } x \in E_{p,q,R} \cap ([-1,0] \times [-1,1]^{n-1}). \tag{40}$$

The rough idea of why this indicates divergence of $T_t f_R$ is the following. If we denote by F the union of all admissible slabs and set the measure $\mu(A) = |A \cap F|/|F|$ supported in F, then we get

$$\big\| \sup_{0<t<1} |T_t f_R| \big\|_{L^1(\mu)} \gtrsim R^{1/4} \left(\frac{R}{DQ}\right)^{\frac{n-1}{2}} \|f_R\|_2, \qquad \forall R \gg 1.$$

Eventually, we will choose D and Q to be powers of R, so we may define the exponent $s(k, \alpha)$ by

$$R^{1/4} \left(\frac{R}{DQ}\right)^{\frac{n-1}{2}} = R^{s(k,\alpha)}. \tag{41}$$

This way, the inequality above morally contradicts the maximal estimate for data with regularity $s < s(\alpha)$, which is the standard way to prove convergence.

However, we will actually prove the divergence property directly. We will sum the f_R above to build a new datum, and we will also build a set F by taking a limsup of the slabs $E_{p,q,R}$ of different q and R. This will be a fractal set whose Hausdorff dimension α will depend on the aforementioned powers D and Q, and where the evolution of the datum, of Sobolev regularity $s < s(k, \alpha)$, will diverge.

As marked, $s(k, \alpha)$ will depend both on the homogeneity degree k and on the Hausdorff dimension chosen α. However, we drop the dependence on k from the notation for simplicity, so we will simply write $s(\alpha)$. We refer to this regularity as the Sobolev exponent.

3.2. **The counterexample.** We just saw how the datum f_R heuristically contradicts the usual maximal estimate. However, it does not give a counterexample for the convergence property. To do that, we combine several f_R at different scales R. We propose

$$f(x) = \sum_{m \geq m_0} \frac{m}{R_m^{s(\alpha)}} \frac{f_{R_m}}{\|f_{R_m}\|_2} \tag{42}$$

for a large enough $m_0 \in \mathbb{N}$, where $R_m = 2^m$ for every $m \geq m_0$. The parameters $D = D(R_m)$ and $Q = Q(R_m)$ corresponding to each component f_{R_m} will be powers of R_m, and we will denote them simply by D_m and Q_m respectively.

We first remark that $f \in H^s(\mathbb{R}^n)$ for every $s < s(\alpha)$, as the triangle inequality shows that

$$\|f\|_s \leq \sum_{m \geq m_0} \frac{m}{R_m^{s(\alpha)-s}} < \infty.$$

In the setting of Proposition 3.8, given that the estimate (40) depends on Q rather than on the particular choice of $q \simeq Q$, we define the sets

$$F_m = \bigcup_{Q_m/2 \leq q \leq Q_m} \bigcup_{p \in G(q)} E_{p,q,R_m}, \qquad m \geq m_0. \tag{43}$$

This means that for every $m \geq m_0$, the component f_{R_m} satisfies (40) in the set F_m.

We now prove the main result of this section, which measures the size of the solution $T_t f$ in the sets F_m.

Proposition 3.9. *Let f be defined in* (42) *and $M \geq m_0$. If $x \in F_M \cap \big([-1,-1/10] \times [-1,1]^{n-1}\big)$, then there exists a time $t(x) \simeq 1/R_M^{k-1}$ such that $\big|T_{t(x)}f(x)\big| \gtrsim M$.*

From this we immediately get the divergence property we are looking for.

Corollary 3.10. *Let $F = \limsup_{m\to\infty} F_m$. Then, for f defined in* (42),

$$\limsup_{t\to 0} \big|T_t f(x)\big| = \infty, \qquad\qquad \forall x \in F \cap \big([-1,-1/10] \times [-1,1]^{n-1}\big).$$

Proof of Corollary 3.10. If $x \in F$, then there are infinitely many $m \in \mathbb{N}$ such that $x \in F_m$. That is, there exists an increasing sequence of natural numbers $(a_n)_{n\in\mathbb{N}}$ such that $x \in F_{a_n}$ for all $n \in \mathbb{N}$. According to Proposition 3.9, for every $n \in \mathbb{N}$ there exist a time $t_n(x)$ such that $t_n(x) \simeq R_{a_n}^{-(k-1)}$ and $|T_{t_n(x)}f(x)| \gtrsim a_n$. Thus, $\lim_{n\to\infty} |T_{t_n(x)}f(x)| = \infty$, which implies the result because $\lim_{n\to\infty} t_n(x) = 0$. □

Let us prove Proposition 3.9. In short, it holds because the main contribution to $T_t f$ in the set F_M comes from the component f_{R_M}, while the effect of the rest of components is negligible.

Proof of Proposition 3.9. Fix $M \in \mathbb{N}$ such that $M \geq m_0$ and let $x \in F_M$. For every $m \geq m_0$, we want to compute the contribution of each of the components $f_{R_m}(x)$. Recall that $x \in F_M$ implies that there are $q \simeq Q_M$ and $p \in G(q)$ such that $x \in E_{p,q,R_M}$, so Proposition 3.8 suggest the choice

$$t = t(x) = p_1/(D_M^k q). \tag{44}$$

We separate cases:

• Case $m = M$. By the choice of t in (44), and recalling the definition of $s(\alpha)$ in (41), Proposition 3.8 implies

$$M\,\frac{|T_t f_{R_M}(x)|}{R_M^{s(\alpha)}\,\|f_{R_M}\|_2} \gtrsim M. \tag{45}$$

• Case $m \neq M$. The objective is to see that the contribution of $T_t f_{R_m}(x)$ for $m \neq M$ is much smaller, so that (45) dominates. Thus, we want to bound $|T_t f_{R_m}(x)|$ from above. From (25) we can directly bound

$$\big|T_t h_{R_m}(x')\big| \leq \sum_{l\in\mathbb{Z}^{n-1}} \psi(D_m l/R_m) \int \big|\widehat{\phi}_2(\xi')\big|\,d\xi' \leq C\left(\frac{R_m}{D_m}\right)^{n-1} \|\widehat{\phi}_2\|_1, \qquad\qquad j = 2,\dots,n,$$

so we write

$$\big|T_t f_{R_m}(x)\big| \lesssim \big|T_t g_{R_m}(x_1)\big| \left(\frac{R_m}{D_m}\right)^{n-1}. \tag{46}$$

Let us analyze the first component. According to (23), we can write

$$\big|T_t g_{R_m}(x_1)\big| = \Big|\int \widehat{\phi}_1(\eta)\,e^{i\,\lambda\,\theta(\eta)}\,d\eta\Big|, \qquad \text{where} \qquad \begin{cases} \lambda = R_m^{1/2}\,\big|x_1 + kR_m^{k-1}t\big|, \\[2ex] \theta(\eta) = \pm\eta + \dfrac{t}{\lambda}\displaystyle\sum_{\ell=2}^{k}\binom{k}{\ell}\,\eta^\ell\,R_m^{k-\ell/2}. \end{cases} \tag{47}$$

and the sing on $\pm\eta$ depends on the sign of $x_1 + tR_m^{k-1}t$. We want to see that

$$\theta'(\eta) = \pm 1 + \frac{t}{\lambda}\sum_{\ell=2}^{k}\binom{k}{\ell}\,\ell\,\eta^{\ell-1}\,R_m^{k-\ell/2} \neq 0 \tag{48}$$

to use the principle of non-stationary phase. Since $x \in E_{p,q,R_M}$ and $x_1 \in [-1,-1/10]$, then the condition $|x_1 + kR_M^{k-1}t| < R_M^{-1/2}$ implies, for $M \geq m_0 \gg 1$, that

$$\frac{1}{R_M^{k-1}} \lesssim \frac{1}{kR_M^{k-1}}\Big(-x_1 - \frac{1}{R_M^{1/2}}\Big) \leq t \leq \frac{1}{kR_M^{k-1}}\Big(-x_1 + \frac{1}{R_M^{1/2}}\Big) \lesssim \frac{1}{R_M^{k-1}}.$$

To see that the second term of $\theta'(\eta)$ in (48) is small, we first bound it by

$$\left| \frac{t}{\lambda} \sum_{\ell=2}^{k} \binom{k}{\ell} \ell\, \eta^{\ell-1} R_m^{k-\ell/2} \right| \leq \frac{tR_m^{k-1}}{\lambda} \sum_{\ell=2}^{k} \ell \binom{k}{\ell} \simeq_k \frac{tR_m^{k-1}}{\lambda} \tag{49}$$

We need to consider two more cases:

- Case $m < M$. In this case $R_m/R_M < 1$, so since $t \simeq 1/R_M^{k-1}$, we can write

$$\left| x_1 + kR_m^{k-1}t \right| = \mathcal{O}(R_M^{-1/2}) + k(R_M^{k-1} - R_m^{k-1})t \simeq 1,$$

so

$$\lambda = R_m^{1/2} \left| x_1 + kR_m^{k-1}t \right| \simeq R_m^{1/2}, \qquad \text{when } m < M, \tag{50}$$

and

$$\frac{tR_m^{k-1}}{\lambda} \simeq \frac{(R_m/R_M)^{k-1}}{R_m^{1/2} \left| x_1 + kR_m^{k-1}t \right|} \lesssim \frac{1}{R_m^{1/2}} < 1/2.$$

- Case $m > M$. Now $R_m/R_M > 1$, so since $t \simeq 1/R_M^{k-1}$, we can write

$$\left| x_1 + kR_m^{k-1}t \right| = \mathcal{O}(R_M^{-1/2}) + k(R_m^{k-1} - R_M^{k-1})t \simeq (R_m/R_M)^{k-1},$$

so

$$\lambda = R_m^{1/2} \left| x_1 + kR_m^{k-1}t \right| \simeq R_m^{1/2} \left(\frac{R_m}{R_M} \right)^{k-1} > R_m^{1/2}, \qquad \text{when } m > M, \tag{51}$$

and

$$\frac{tR_m^{k-1}}{\lambda} \simeq \frac{1}{R_m^{1/2}} < 1/2.$$

In both cases we get $tR_m^{k-1}/\lambda < 1/2$, so from (48) and (49) we deduce that $|\theta'(\eta)| > 1/2$. This allows us to integrate (47) by parts as many times as we need to obtain the bound

$$|T_t g_{R_m}(x_1)| \lesssim_N \frac{1}{\lambda^N}, \qquad \text{for all } N \in \mathbb{N}.$$

In (50) and (51) we got $\lambda \gtrsim R_m^{1/2}$ for every $m \neq M$, so $|T_t g_{R_m}(x_1)| \lesssim_N R_m^{-N/2}$ for every $m \neq M$. Together with (41) and (46), this implies

$$m \frac{|T_t f_{R_m}(x)|}{R_m^{s(\alpha)} \, \|f_{R_m}\|_2} \lesssim m \frac{R_m^{1/4}}{R_m^{s(\alpha)}} \left(\frac{R_m}{D_m} \right)^{\frac{n-1}{2}} \frac{1}{R_m^{N/2}} = \frac{m\, Q_m^{\frac{n-1}{2}}}{R_m^{N/2}} < \frac{1}{R_m}, \qquad \text{for all } m \neq M, \tag{52}$$

where the last inequality is true if we choose N as large as we need (recall that Q_m is a some power of R_m). Finally, joining (45) and (52), for $x \in F_M$ we have found a time $t = t(x)$ in (44) such that

$$\begin{aligned}
\left| T_{t(x)} f(x) \right| &= \left| \sum_{m \geq m_0} m \frac{T_t f_{R_m}(x)}{R_m^{s(\alpha)} \, \|f_{R_m}\|_2} \right| \geq M \frac{|T_t f_{R_M}(x)|}{R_M^{s(\alpha)} \, \|f_{R_M}\|_2} - \sum_{m \neq M} m \frac{|T_t f_{R_m}(x)|}{R_m^{s(\alpha)} \, \|f_{R_m}\|_2} \\
&\geq M - \sum_{m \neq M} \frac{1}{R_m} \geq M - \sum_{m=1}^{\infty} \frac{1}{2^m} \\
&\gtrsim M,
\end{aligned}$$

the last inequality being true if $M \geq m_0$ and m_0 is large enough.

Incidentally, our computations show that f is essentially equal to f_{R_M} around the plane $\{(x,t) \mid x_1 + kR_M^{k-1}t = 0\}$ when $x \in [-1,-1/10] \times [-1,1]^{n-1}$, so f is continuous and well defined everywhere around that plane — recall our convention (3). $\square$

To conclude the proof of Theorem 1.4, we are missing to compute the Hausdorff dimension of the set of divergence, $F = \limsup_{m\to\infty} F_m$. We do that using the Mass Transference Principle.

4. The Mass Transference Principle

The Mass Transference Principle was introduced in [4] in the setting of the Duffin-Schaeffer conjecture, which has recently been solved, as a method to compute the α-Hausdorff measure of limsup sets. Let $x \in \mathbb{R}^n$ and $r > 0$, and denote by $B = B(x,r)$ the ball with center at x and radius r. Let $0 \leq a < 1$ and define the dilated ball $B^a = B(x, r^a)$.

Theorem 4.1 (Mass Tranference Principle, Theorem 2 in [4])**.** *Let $\{B_i\}_{i\in\mathbb{N}}$ be a sequence of balls in $\mathbb{R}^n$. Assume that the radii $r_i > 0$ of B_i satisfy $\lim_{i\to\infty} r_i = 0$, and suppose that*

$$\mathcal{H}^n\left(B \cap \limsup_{i\to\infty} B_i^{s/n}\right) = \mathcal{H}^n(B), \qquad \textit{for every ball } B \subset \mathbb{R}^n. \tag{53}$$

Then,

$$\mathcal{H}^s\left(B \cap \limsup_{i\to\infty} B_i\right) = \mathcal{H}^s(B), \qquad \textit{for every ball } B \subset \mathbb{R}^n.$$

Observe that the hypothesis (53) means that $\mathcal{H}^n(\mathbb{R}^n \setminus \limsup_{i\to\infty} B_i^{s/n}) = 0$, that is, that the set $\limsup_{i\to\infty} B_i^{s/n}$ has full measure in $\mathbb{R}^n$. On the other hand, if $s < n$, every ball $B \subset \mathbb{R}^n$ satisfies $\mathcal{H}^s(B) = \infty$. Thus, the conclusion of Theorem 4.1 is that $\mathcal{H}^s(\limsup_{i\to\infty} B_i) = \infty$, which in turn implies that $\dim(\limsup_{i\to\infty} B_i) \geq s$. This means that if we can rescale the balls so that their limsup has full Lebesgue measure, the mass transference principle gives a lower bound for the Hausdorff dimension of the limsup of the original balls.

Theorem 4.1 is a powerful tool to obtain deep results by very simple computations. As an example, let us explain how the celebrated Jarník-Besicovitch theorem [20, 21] can be easily obtained using the mass transference principle and the Dirichlet approximation theorem.

In Diophantine approximation, a very well-known result (consequence either of the Dirichlet approximation theorem or of the theory of continued fractions) is that every real number can be approximated by infinitely many rational numbers p/q with an error smaller than q^{-2}. In other words,

$$\left\{x \in \mathbb{R} \,:\, \left|x - \frac{p}{q}\right| < \frac{1}{q^2} \text{ for infinitely many rationals } \frac{p}{q}\right\} = \mathbb{R}. \tag{54}$$

One may wonder how much better than q^{-2} the error can be made, so it is natural to study how large the sets

$$S_\tau = \left\{x \in [0,1] \,:\, \left|x - \frac{p}{q}\right| < \frac{1}{q^\tau} \text{ for infinitely many rationals } \frac{p}{q}\right\}, \qquad \tau \geq 2, \tag{55}$$

are. The Jarník-Besicovitch theorem answers this question.

Theorem 4.2 (Jarník-Besicovitch theorem)**.** *Let $\tau \geq 2$ and S_τ be defined in (55). Then,*

$$\dim S_\tau = 2/\tau.$$

As usual, the upper bound is a direct consequence of the covers $S_\tau \subset \bigcup_{q\geq Q} B\left(p/q, 1/q^\tau\right)$ for all $Q\in\mathbb{N}$. If $s>2/\tau$, they yield

$$\mathcal{H}^s_{1/Q^\tau}(S^\tau) \leq \sum_{q\geq Q} \frac{q}{q^{\tau s}} = \sum_{q\geq Q} \frac{1}{q^{\tau s-1}} \to 0 \qquad \text{when } Q\to\infty,$$

so $\dim S_\tau \leq 2/\tau$. The lower bound, independently obtained by Jarník [20, 21] and Besicovitch [5], is not that easy to prove by hand. Let us use Theorem 4.1 instead. Recall first that the limsup of a sequence of sets $(F_j)_{j\in\mathbb{N}}$ is defined as $\limsup_{j\to\infty} F_j = \bigcap_{J\in\mathbb{N}}\bigcup_{j\geq J} F_j$, so it can be characterized as

$$x \in \limsup_{j\to\infty} F_j \qquad \Longleftrightarrow \qquad \exists \text{ infinitely many } j\in\mathbb{N} \text{ such that } x\in F_j.$$

For the sets S_τ, it suffices to work with rationals $p/q\in[0,1]$, which can be ordered increasingly in q given that $0\leq p\leq q$. Thus, S_τ is the limsup of the balls $B(p/q,1/q^\tau)$. Now, by (54) we know that S_2 has full Lebesgue measure, and

$$\mathcal{H}^n\left(\limsup_{\substack{q\to\infty\\0\leq p\leq q}} B\left(\frac{p}{q},\frac{1}{q^\tau}\right)^{2/\tau}\right) = \mathcal{H}^n\left(\limsup_{\substack{q\to\infty\\0\leq p\leq q}} B\left(\frac{p}{q},\frac{1}{q^2}\right)\right) = \mathcal{H}^n\left(S_2\right) = 1.$$

Theorem 4.1 implies that

$$\mathcal{H}^{2/\tau}\left(S_\tau\right) = \mathcal{H}^{2/\tau}\left(\limsup_{\substack{q\to\infty\\0\leq p\leq q}} B\left(\frac{p}{q},\frac{1}{q^\tau}\right)\right) = \infty$$

and thus $\dim S_\tau \geq 2/\tau$.

In a similar way, we will use the mass transference principle to compute the Hausdorff dimension of the divergence set of $T_t f$. However, Theorem 4.1 employs a transition from balls to balls, while our sets are limsups of rectangles, not balls. But the mass transference principle has been adapted to tackle transitions from balls to rectangles in [41] and from rectangles to rectangles recently in [40]. We make use of the latter. For that, we need to adapt our setting to the framework in [40, Section 3.1]. For the sake of comparison, we use the same notation as there.

Recall that our divergence set F is the limsup of the sets F_m in (43). To define a numbering J of the slabs in F, we number the slabs within each F_m, so we index each slab in F as $\alpha=(m,\nu)$, where ν refers to the numbering within F_m. The resonant sets $\{\mathcal{R}_\alpha \mid \alpha\in J\}$ are thus the centers of the admissible slabs (39), so the scaling property is $\kappa=0$ (see [40, Definition 3.1]).

Also following [40], we define the function $\beta : J\to\mathbb{R}_+$ as $\beta(m,i)=2^m=R_m$ and the functions $u_m=l_m=\beta(m,i)$ so that $J_m=\{\alpha\in J \mid l_m\leq\beta(\alpha)\leq u_m\}$ index the slabs in F_m. If we define $\rho(u)=u^{-1}$, then our slabs can be written as

$$E_{p,q,R_m} = B(\mathcal{R}_\alpha,\rho(R_m))^{\boldsymbol{b}} = \prod_{i=1}^n B\left((\mathcal{R}_\alpha)_i,\rho(R_m)^{b_i}\right),$$

where $\boldsymbol{b}=(1/2,1,\ldots,1)$.

The following is [40, Definition 3.3], which means that the dilated rectangles $B(\mathcal{R}_\alpha,\rho(R_m))^{\boldsymbol{a}}$, with $\boldsymbol{a}=(a_1,\ldots,a_n)$ and $a_i\leq b_i$, fill the space.

Definition 4.3 (Uniform local ubiquity for rectangles)**.** A system $\{R_\alpha \mid \alpha\in J\}$ is uniformly locally ubiquitous with respect to $(\rho,\boldsymbol{a})$ if there exists a constant $c>0$ such that for any ball B

$$\mathcal{H}^n\Big(B\cap\bigcup_{\alpha\in J_m} B(\mathcal{R}_\alpha,\rho(R_m))^{\boldsymbol{a}}\Big) \geq c\,\mathcal{H}^n(B) \qquad \text{for all } m\geq m_0(B). \tag{56}$$

Uniform local ubiquity is actually stronger than full measure, and it can be used to get some refinements of the mass transference principle, but we do not exploit them to avoid unnecessary technical details.

We state now an adapted version of Theorem 3.1 of [40] considering the simplification in Proposition 3.1 there.

Theorem 4.4 (Mass transference principle from rectangles to rectangles)**.** *In $\mathbb{R}^n$, let $\{\mathcal{R}_\alpha \mid \alpha \in J\}$ be a uniformly locally ubiquitous system with respect to $(\rho, \boldsymbol{a})$. Let $\boldsymbol{b} = (b_1, \ldots, b_n)$ be an exponent with $b_i \geq a_i$ for $i = 1, \ldots, n$, and define the set*

$$W(\boldsymbol{b}) = \Big\{ x \in \mathbb{R}^n \, : \, x \in B(\mathcal{R}_\alpha, \rho(R_m))^{\boldsymbol{b}} \text{ for infinitely many } \alpha \in J \Big\}.$$

Then,

$$\dim W(\boldsymbol{b}) \geq \min_{A \in \mathcal{A}} \left\{ \sum_{j \in K_1(A)} 1 + \sum_{j \in K_2(A)} \left(1 - \frac{b_j - a_j}{A} \right) + \sum_{j \in K_3(A)} \frac{a_j}{A} \right\}, \tag{57}$$

where $\mathcal{A} = \{b_1, \ldots, b_n\}$ and for every $A \in \mathcal{A}$, the sets $K_1(A), K_2(A)$ and $K_3(A)$ form a partition of $\{1, \ldots, n\}$ in the following way:

$$K_1(A) = \{j \, : \, a_j \geq A\}, \quad K_2(A) = \{j \, : \, b_j \leq A\} \setminus K_1(A), \quad K_3(A) = \{1, \ldots, n\} \setminus (K_1(A) \cup K_2(A)).$$

Remark 4.5. *Like the original mass transference principle in Theorem 4.1, one can interpret this result in terms of dilations. Denote $\boldsymbol{a}/\boldsymbol{b} = (a_1/b_1, \ldots a_n/b_n)$ so that $B(\mathcal{R}_\alpha, \rho)^{\boldsymbol{a}} = \big(B(\mathcal{R}_\alpha, \rho)^{\boldsymbol{b}} \big)^{\boldsymbol{a}/\boldsymbol{b}}$. Thus, if we dilate the original balls $B(\mathcal{R}_\alpha, \rho)^{\boldsymbol{b}}$ with a dilation exponent $\boldsymbol{s} = \boldsymbol{a}/\boldsymbol{b}$ to obtain balls $B(\mathcal{R}_\alpha, \rho)^{\boldsymbol{a}}$ whose limsup has full Lebesgue measure, then the dimension of $W(\boldsymbol{b})$ is given by (57), which can be written in terms of $\boldsymbol{b}$ and $\boldsymbol{s}$ only.*

Remark 4.6. *Theorem 4.1 is essentially recovered by setting $\boldsymbol{b} = (1, \ldots, 1)$ and $\boldsymbol{a} = (a, \ldots, a)$, with $a < 1$, so that $K_2(1) = \{1, \ldots, n\}$ and $\dim W(\boldsymbol{b}) \geq na = s$. Indeed, $B^{s/n} = B^a$.*

5. The set of divergence and its Hausdorff dimension

We now apply the mass transference principle in Theorem 4.4 to compute the Hausdorff dimension of the divergence set $F = \limsup_{m \to \infty} F_m$.

5.1. **The set and the geometric parameters.** Let $R > 1$ and $D, Q \geq 1$ be our usual two parameters, which are powers of R. Based on the slabs (39), we define

$$\begin{aligned} F_R &= \bigcup_{Q/2 \leq q \leq Q} \, \bigcup_{p \in G(q)} E_{p,q,R} \\ &= \bigcup_{Q/2 \leq q \leq Q} \, \bigcup_{p \in G(q)} B_1 \left(k \, \frac{R^{k-1}}{D^k} \frac{p_1}{q}, \frac{1}{R^{1/2}} \right) \times B_{n-1} \left(\frac{1}{D} \frac{p'}{q}, \frac{1}{R} \right). \end{aligned} \tag{58}$$

The sets F_m in (43) that give rise to the divergence set F correspond to $F_m = F_{R_m}$ for the particular choice of $R = R_m = 2^m$. As we will see, the Hausdorff dimension of F depends on the choice of D and Q. In short, to use the mass transference principle in Theorem 4.4, we need to find a dilation of the slabs in (58) such that the limsup of the dilated slabs cover the whole $[-1, 1]^n$ (or, more precisely, such that they satisfy (56)).

For that purpose, it is convenient to replace D and Q with two new parameters that control the separation between slabs. For each $q \simeq Q$, the separation in the coordinate x_1 of two consecutive slabs in (58) is $kR^{k-1}/(D^kQ)$, so let us define the parameter u_1 by

$$R^{u_1} = \frac{D^kQ}{R^{k-1}}. \tag{59}$$

In the same way, the separation in each of the remaining coordinates x_j is $1/(DQ)$, so we define

$$R^{u_2} = DQ. \tag{60}$$

Direct computation shows that (59) and (60) can be reversed as

$$D = R^{1-\frac{u_2-u_1}{k-1}}, \qquad\qquad Q = R^{\frac{ku_2-u_1}{k-1}-1}. \tag{61}$$

We have thus a one to one correspondence between the pairs (D, Q) and (u_1, u_2). Even if (D, Q) are natural to build the counterexample (21), (u_1, u_2) are more convenient to work geometrically.

5.2. **Basic restrictions for the parameters.** The new parameters (u_1, u_2) cannot be arbitrary. In this subsection we determine the basic restrictions they have to satisfy and delimit their domain in the plane.

First, for each q, the separation in x_1 should be small enough so that there is more than a single slab in $[0, 1]$. For that, we ask

$$\frac{1}{R^{u_1}} \leq 1 \qquad \Longrightarrow \qquad u_1 \geq 0. \tag{62}$$

Also, the slabs will not intersect in direction x_1 if we ask

$$\frac{1}{R^{1/2}} \leq \frac{1}{R^{u_1}} \quad \Longrightarrow \quad u_1 \leq 1/2. \tag{63}$$

Analogously, in the coordinates $x_2, \ldots, x_n$ we require

$$\frac{1}{R^{u_2}} \leq 1 \quad \Longrightarrow \quad u_2 \geq 0 \qquad \text{and} \qquad \frac{1}{R} \leq \frac{1}{R^{u_2}} \quad \Longrightarrow \quad u_2 \leq 1. \tag{64}$$

One more restriction comes from the condition $Q \geq 1$, which by (61) implies

$$\boxed{ku_2 - u_1 \geq k - 1.} \tag{65}$$

Inequalities (62), (63), (64) and (65) determine the basic region of validity for (u_1, u_2), which we show in Figure 4. We remark that, in particular, we have

$$u_2 \geq \frac{u_1}{k} + \frac{k-1}{k} \geq \frac{k-1}{k} \geq \frac{1}{2} \tag{66}$$

as a consequence of (62) and $k \geq 2$.

To apply the mass transference principle, we need to impose further restrictions on (u_1, u_2).

5.3. **Preparing the setting for the Mass Transference Principle.** We now turn to the setting of the mass transference principle in Theorem 4.4. Recall from (58) that the slabs are

$$E_{p,q,R} = B_1\left(k\,\frac{R^{k-1}}{D^k}\frac{p_1}{q}, \frac{1}{R^{1/2}}\right) \times B_{n-1}\left(\frac{1}{D}\frac{p'}{q}, \frac{1}{R}\right). \tag{67}$$

As we mentioned in Section 4, the radius parameter $\rho(R) = 1/R \to 0$, so we need:

- the exponent
$$\boldsymbol{b} = (b_1, b_2, \ldots, b_2) = (1/2, 1, \ldots, 1),$$
corresponding to the radii of the original slabs (67).

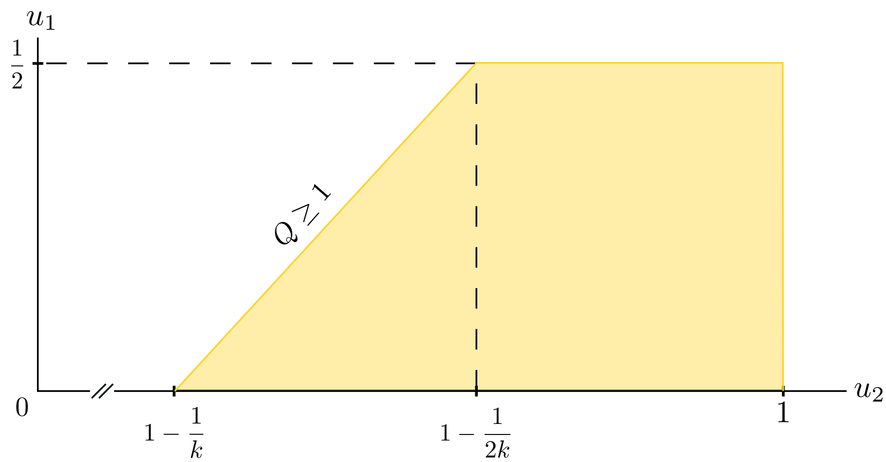


FIGURE 4. In yellow, the basic trapezoidal region for the parameters (u_1, u_2).

- the exponent
$$\boldsymbol{a} = (a_1, a_2, \ldots, a_2)$$
so that the limsup of the dilated slabs
$$E_{p,q,R,\boldsymbol{a}} = B_1\left(k\,\frac{R^{k-1}}{D^k}\frac{p_1}{q}, \frac{1}{R^{a_1}}\right) \times B_{n-1}\left(\frac{1}{D}\,\frac{p'}{q}, \frac{1}{R^{a_2}}\right) \tag{68}$$
satisfies the uniform local ubiquity condition in Definition 4.3. In other words, for any ball B, we have to show that the Lebesgue measure of the set
$$F_{R,\boldsymbol{a}} \cap B := \bigcup_{Q/2\le q\le Q}\ \bigcup_{p\in G(q)} E_{p,q,R,\boldsymbol{a}} \cap B$$
is large.

The set $F_{R,\boldsymbol{a}}$ has a periodic structure. It is made up of translations of the unit cell
$$\widetilde{F}_{R,\boldsymbol{a}} := \bigcup_{Q/2\le q\le Q}\ \bigcup_{p\in G(q)\cap[0,q)^n} E_{p,q,R,\boldsymbol{a}},$$
as shown on the left hand side of Figure 5. A ball B contains approximately $\mathcal{H}^n(B)\,D^{n-1}\,D^k/(kR^{k-1})$ translated and disjoint copies of $\widetilde{F}_{R,\boldsymbol{a}}$ as long as $R \gg 1$ and D satisfies
$$\frac{D^k}{R^{k-1}} > 1, \qquad \text{which implies} \qquad \boxed{u_2 - u_1 < 1 - \frac{1}{k}.} \tag{69}$$
Let us define the transformation $T(x) = (\frac{D^k}{k\,R^{k-1}}\,x_1, Dx')$. Then, $T(\widetilde{F}_{R,\boldsymbol{a}}) = \Omega_{R,\boldsymbol{a}}$, where
$$\Omega_{R,\boldsymbol{a}} = \bigcup_{Q/2\le q\le Q}\ \bigcup_{p\in G(q)\cap[0,q)^n} B_1\left(\frac{p_1}{q}, \frac{D^k}{k\,R^{k-1}\,R^{a_1}}\right) \times B_{n-1}\left(\frac{p'}{q}, \frac{D}{R^{a_2}}\right)$$
is shown on the right hand side of Figure 5. By the scaling properties of the Lebesgue measure,
$$\mathcal{H}^n\big(\widetilde{F}_{R,\boldsymbol{a}}\big) = \frac{kR^{k-1}}{D^k}\,\frac{1}{D^{n-1}}\,\mathcal{H}^n(\Omega_{R,\boldsymbol{a}}),$$
so $\mathcal{H}^n(F_{R,\boldsymbol{a}} \cap B)/\mathcal{H}^n(B) \simeq \mathcal{H}^n(\Omega_{R,\boldsymbol{a}})$. Thus, to check whether the dilated slabs $E_{p,q,R,\boldsymbol{a}}$ form a uniformly locally ubiquitous system, it is enough to prove that $\mathcal{H}^n(\Omega_{R,\boldsymbol{a}}) \ge c > 0$.

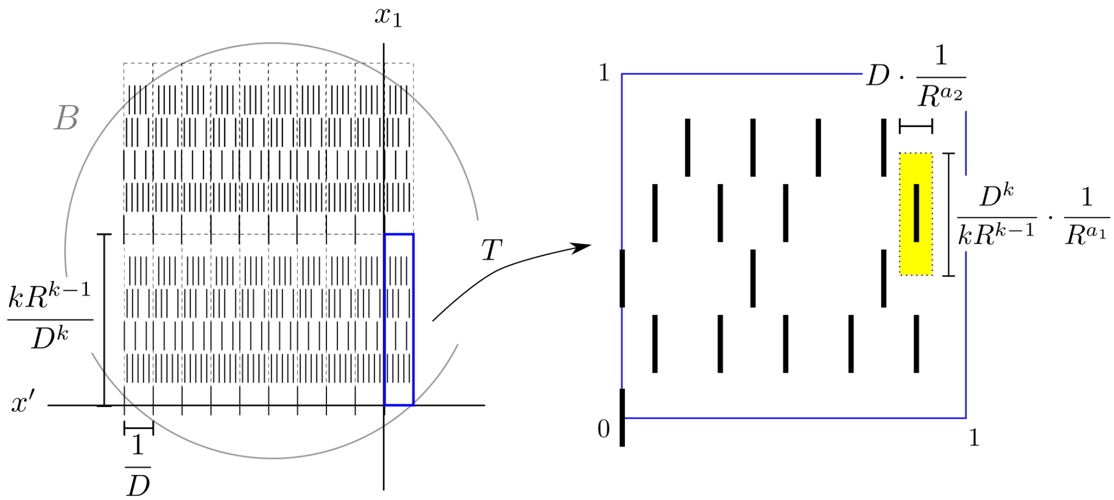


FIGURE 5. At the left, the union F_R of admissible slabs, which has a periodic structure. The unit cell $\widetilde{F}_{R,\boldsymbol{a}}$ is shown in blue. At the right, the image $T(\widetilde{F}^k_{R,\boldsymbol{a}}) = \Omega_{R,\boldsymbol{a}}$ of the unit cell by the transformation T. In yellow, the image $T(E_{p,q,R,\boldsymbol{a}})$ of a dilated admissible slab in (68). To use the mass transference principle, the set $\Omega_{R,\boldsymbol{a}}$, which is the union of the dilated admissible slabs, should cover a positive portion of the unit cell.

We now give some basic restrictions for $\boldsymbol{a}$. First, we need that the dilated slabs (68) are contained in $[-1,1]^n$, so

$$\frac{1}{R^{a_1}}, \frac{1}{R^{a_2}} \leq 1 \qquad \Longrightarrow \qquad a_1, a_2 \geq 0. \tag{70}$$

Also, we want the dilation (68) to be larger than the original slab (67), so

$$\frac{1}{R^{a_1}} \geq \frac{1}{R^{1/2}} \quad \Longrightarrow \quad a_1 \leq 1/2, \qquad \text{and} \qquad \frac{1}{R^{a_2}} \geq \frac{1}{R} \quad \Longrightarrow \quad a_2 \leq 1. \tag{71}$$

5.4. **Conditions for uniform local ubiquity.** We now look for the conditions on $\boldsymbol{a}$ so that $\mathcal{H}^n(\Omega_{R,\boldsymbol{a}}) \geq c > 0$. In [1, Section 4], they found conditions that yield the slightly weaker result $\mathcal{H}^n(\Omega_{R,\boldsymbol{a}}) \geq 1/\log Q$. In the following lines, we avoid the logarithmic loss by making the dilated slabs $E_{p,q,R,\boldsymbol{a}}$ an ϵ larger, and hence the dilation exponent $\boldsymbol{a}$ an ϵ smaller. This eventually results in a loss of an ϵ in the Hausdorff dimension of F, which will not affect the final result. We need the following auxiliary lemma, which is [1, Lemma 4.1].

Lemma 5.1 (Lemma 4.1 of [1])**.** *Let J be a finite set of indices and $\{I_j\}_{j\in J}$ be a collection of measurable sets in $\mathbb{R}^n$. Suppose that these sets have comparable sizes, that is, $B_0 \leq |I_j| \leq B_1$ for every $j \in J$, and that they are regularly distributed in the sense that*

$$|\{(j,j') \in J \quad : \quad I_j \cap I_j \neq \emptyset\}| \leq C_1\,|J|. \tag{72}$$

Then,

$$|\bigcup_{j\in J} I_j| \geq \frac{B_0}{B_1\,C_1} \sum_{j\in J} |I_j|.$$

With this lemma, we can prove the following proposition, which is an adapted version of [1, Proposition 4.2] and should be compared with Minkowski's theorem on intersections of lattice points with a convex, symmetric body.

Proposition 5.2. *Let R, D, Q be the parameters of our problem. Define $\mathcal{P}_Q = \{q \in [Q/2, Q] :$ q prime $\}$, and let $G(q) \subset [0,q)^n$ satisfy $|G(q)| \simeq q^n$ for every $q \in \mathcal{P}_Q$. Suppose there exist $0 < E_1, E_2, E_3, E_4 < 1$ such that*

$$\frac{E_1}{x^\alpha} \le h_1(x) \le \frac{E_2}{x^\alpha} \qquad \text{and} \qquad \frac{E_3}{x^\beta} \le h_2(x) \le \frac{E_4}{x^\beta}$$

with $\alpha, \beta \ge 1$. If

$$\Omega = \bigcup_{q \in P_Q} \bigcup_{p \in G(q)} B_1\left(\frac{p_1}{q}, h_1(Q)\right) \times B_{n-1}\left(\frac{p'}{q}, h_2(Q)\right),$$

then

$$\mathcal{H}^n(\Omega) \gtrsim \frac{|\mathcal{P}_Q|\, Q^n\, h_1(Q) h_2(Q)^{n-1}}{1 + |\mathcal{P}_Q|\, Q^n\, h_1(Q) h_2(Q)^{n-1}}.$$

In particular, if $h_1(Q)\, h_2(Q)^{n-1} \simeq 1/Q^{n+1-\epsilon}$ for $\epsilon > 0$, then

$$\mathcal{H}^n(\Omega) \ge c > 0.$$

Remark 5.3. *Proposition 4.2 of [1] corresponds to asking $|\mathcal{P}_Q|\, Q^n\, h_1(Q) h_2(Q)^{n-1} \lll 1$. Indeed, taking into account that $|\mathcal{P}_Q| \simeq Q/\log Q$, it amounts to asking $h_1(Q) h_2(Q)^{n-1} \lll \log Q/Q^{n+1}$. Given that $h_1(Q) \le 1/Q$, then it would be enough to have $h_2(Q)^{n-1} \lll \log Q/Q^n$, and in particular, $h_2(Q) \le 1/Q^{n/(n-1)}$. Under those conditions, $\mathcal{H}^n(\Omega) \gtrsim |\mathcal{P}_Q|\, Q^n\, h_1(Q) h_2(Q)^{n-1}$.*

Proof. We use Lemma 5.1. For simplicity, let us call

$$I_{p,q} = B_1\left(\frac{p_1}{q}, h_1(Q)\right) \times B_{n-1}\left(\frac{p'}{q}, h_2(Q)\right) \quad \text{so that} \quad \Omega = \bigcup_{q \in \mathcal{P}_Q} \bigcup_{p \in G(q)} I_{p,q}. \tag{73}$$

All the slabs have the same measure $|I_{p,q}| = h_1(Q)\, h_2(Q)^{n-1}$, so $B_0 = B_1$. Since the set of indices in (73) has size $\sum_{q \in \mathcal{P}_Q} |G(q)| \simeq |\mathcal{P}_Q|\, Q^n$, to verify (72) we need to prove

$$\{(p,q), (\tilde{p}, \tilde{q}) \in G(q) \times \mathcal{P}_Q \,:\, I_{p,q} \cap I_{\tilde{p},\tilde{q}} \ne \emptyset\} \le C_1\, |\mathcal{P}_Q|\, Q^n. \tag{74}$$

We separate into three cases:

• **Case 1.** When the tuples are identical $(p,q) = (\tilde{p}, \tilde{q})$, the condition $I_{p,q} \cap I_{\tilde{p},\tilde{q}} \ne \emptyset$ is obviously satisfied. There are $\sum_{q \in \mathcal{P}_Q} |G(q)| \simeq |\mathcal{P}_Q|\, Q^n$ such tuples.

The two other cases concern non identical tuples. In this case, $I_{p,q} \cap I_{\tilde{p},\tilde{q}} \ne \emptyset$ implies

$$\left|\frac{p_1}{q} - \frac{\tilde{p}_1}{\tilde{q}}\right| \le h_1(Q) \qquad \text{and} \qquad \left|\frac{p}{q} - \frac{\tilde{p}_j}{\tilde{q}}\right| \le h_2(Q), \quad j = 2, \dots, n. \tag{75}$$

• **Case 2.** When tuples are non-identical with $q = \tilde{q}$, then from (75)

$$|p_1 - \tilde{p}_1| \le q h_1(Q) \le E_2 q/Q^\alpha < 1$$

because $\alpha \ge 1$. That implies $p_1 = \tilde{p}_1$. In the same way,

$$|p_j - \tilde{p}_j| \le q h_2(Q) \le E_2 q/Q^\beta < 1, \qquad j = 2, \dots, n,$$

because $\beta \ge 1$. Thus, $p_\ell = \tilde{p}_\ell$ for $\ell = 2, \dots, n$. This means that $p = \tilde{p}$, so there are no tuples in this case.

• **Case 3.** When tuples are non-identical with $q \ne \tilde{q}$, then from (75) we get

$$|\tilde{q} p_1 - q \tilde{p}_1| \le q \tilde{q} h_1(Q) \le Q^2\, h_1(Q) \quad \text{and} \quad |\tilde{q} p_\ell - q \tilde{p}_\ell| \le q \tilde{q} h_2(Q) \le Q^2\, h_2(Q), \quad \ell = 2, \dots, n. \tag{76}$$

Fix $q \ne \tilde{q}$ and $\ell \in \{1, \dots, n\}$.

In how many ways can we represent a given integer m as $\tilde{q} p_\ell - q \tilde{p}_\ell$ with $0 \le p_\ell < q$ and $0 \le \tilde{p}_\ell < \tilde{q}$? For that, assume $\tilde{q} r_j - q \tilde{r}_j$ is other representation of m so that $\tilde{q} p_\ell - q \tilde{p}_\ell = \tilde{q} r_\ell - q \tilde{r}_\ell$.

Then, $\tilde{q}(p_\ell - r_\ell) = q(\tilde{p}_\ell - \tilde{r}_\ell)$. Since $q, \tilde{q} \in \mathcal{P}_Q$ are coprime, q must divide $p_\ell - r_\ell$ and necessarily $p_\ell = r_\ell$. Hence, the representation of m is unique.

Since each integer m with $|m| \leq Q^2 h_1(Q)$ has at most one representation $\tilde{q}p_j - q\tilde{p}_j$, it means that there are at most $2Q^2 h_1(Q) + 1$ pairs $(p_1, \tilde{p}_1)$ that can satisfy (76). Analogously, for $\ell = 2, \ldots, n$ and for an integer m with $|m| \leq Q^2 h_2(Q)$ there is at most one pair $(p_\ell, \tilde{p}_\ell)$ that represent m. Thus, there are at most $2Q^2 h_2(Q) + 1$ pairs $(p_\ell, \tilde{p}_\ell)$ that can satisfy (76). In all, there are at most $(2Q^2 h_1(Q) + 1)(2Q^2 h_2(Q) + 1)^{n-1} \leq 4^n\, Q^{2n} h_1(Q) h_2(Q)^{n-1}$ pairs $(p, \tilde{p})$ that can satisfy (76).

Finally, since $q, \tilde{q} \in \mathcal{P}_Q$, there are at most $4^n\, |\mathcal{P}_Q|^2 Q^{2n} h_1(Q) h_2(Q)^{n-1}$ pairs of tuples $(p, q), (\tilde{p}, \tilde{q})$ in case 3.

We now join the three cases to see that

$$\begin{aligned}\{(p,q),(\tilde{p},\tilde{q}) \in G(q) \times \mathcal{P}_Q \,:\, I_{p,q} \cap I_{\tilde{p},\tilde{q}} \neq \emptyset\} &\lesssim |\mathcal{P}_Q|\, Q^n + |\mathcal{P}_Q|^2 Q^{2n} h_1(Q) h_2(Q)^{n-1} \\ &= |\mathcal{P}_Q|\, Q^n \left(1 + |\mathcal{P}_Q| Q^n h_1(Q) h_2(Q)^{n-1}\right).\end{aligned}$$

Then, (74) is satisfied with $C_1 = 1 + |\mathcal{P}_Q| Q^n h_1(Q) h_2(Q)^{n-1}$ and thus Lemma 5.1 implies

$$\mathcal{H}^n(\Omega) \gtrsim \frac{\sum_{q \in \mathcal{P}_Q} \sum_{p \in G(q)} h_1(Q) h_2(Q)^{n-1}}{1 + |\mathcal{P}_Q|\, Q^n\, h_1(Q) h_2(Q)^{n-1}} \simeq \frac{|\mathcal{P}_Q|\, Q^n\, h_1(Q) h_2(Q)^{n-1}}{1 + |\mathcal{P}_Q|\, Q^n\, h_1(Q) h_2(Q)^{n-1}}. \tag{77}$$

Let now $\epsilon > 0$ and assume that $h_1(Q) h_2(Q)^{n-1} \simeq 1/Q^{n+1-\epsilon}$. That means that

$$|\mathcal{P}_Q|\, Q^n\, h_1(Q) h_2(Q)^{n-1} \simeq \frac{Q^{n+1}}{\log Q} \frac{1}{Q^{n+1-\epsilon}} = \frac{Q^\epsilon}{\log Q} \gg 1,$$

so (77) implies $\mathcal{H}^n(\Omega) \geq c > 0$. $\square$

Coming back to the set $\Omega_{R,\boldsymbol{a}}$, and having in mind that both Q and D are powers of R, we have

$$h_1(Q) = \frac{D^k}{kR^{k-1} R^{a_1}} \qquad \text{and} \qquad h_2(Q) = \frac{D}{R^{a_2}}.$$

Proposition 5.2 shows that $|\Omega_{R,\boldsymbol{a}}| \geq c > 0$ (and consequently that the dilated slabs form a uniform local ubiquity system) whenever the following three conditions hold:

$$(i) \quad \frac{D^k}{kR^{k-1} R^{a_1}} \leq \frac{1}{Q}; \qquad \text{and} \qquad (ii) \quad \frac{D}{R^{a_2}} \leq \frac{1}{Q};$$

and, for $\epsilon > 0$,

$$(iii) \quad \frac{D^k}{R^{k-1} R^{a_1}} \left(\frac{D}{R^{a_2}}\right)^{n-1} = \frac{1}{Q^{n+1-\epsilon}} \qquad \Longleftrightarrow \qquad Q^{1-\epsilon}\, R^{u_1}\, R^{(n-1)u_2} \simeq R^{a_1 + (n-1)a_2}.$$

In view of the definition of (u_1, u_2) in (59), (60) and (61), comparing the exponents we get

$$(i) \quad a_1 \geq u_1; \qquad \text{and} \qquad (ii) \quad a_2 \geq u_2; \tag{78}$$

and, for $\epsilon > 0$,

$$(iii) \quad a_1 + (n-1)a_2 = \frac{k - 2 + \epsilon}{k-1}\, u_1 + \frac{n(k-1) + 1 - k\epsilon}{k-1}\, u_2 - (1 - \epsilon). \tag{79}$$

Together with (70) and (71), these complete the restrictions for the exponent $\boldsymbol{a}$. We gather all in the following lemma.

Lemma 5.4. *Fix the parameters (u_1, u_2) and $\epsilon > 0$. If $R \gg 1$ and if the dilation exponent $\boldsymbol{a} = (a_1, a_2)$ satisfies $u_1 \leq a_1 \leq 1/2$, $u_2 \leq a_2 \leq 1$ and*

$$a_1 + (n-1)a_2 = \frac{k - 2 + \epsilon}{k-1}\, u_1 + \frac{n(k-1) + 1 - k\epsilon}{k-1}\, u_2 - (1 - \epsilon), \tag{80}$$

then the dilated admissible slabs (68) form a uniformly locally ubiquitous system as in Definition 4.3.

Remark 5.5. *This lemma means that, for fixed* (u_1, u_2), (a_1, a_2) *must be chosen from the intersection of the line* (80) *with the rectangle* $[u_1, 1/2] \times [u_2, 1]$. *Many times, it will be useful to rewrite* (80) *as*

$$\frac{k-2}{k-1}\,u_1 + \frac{n(k-1)+1}{k-1}\,u_2 = a_1 + (n-1)a_2 + 1 + \epsilon\left(\frac{ku_2 - u_1}{k-1} - 1\right). \tag{81}$$

To simplify the reading in the forthcoming sections, we denote this domain by $\mathcal{A}$, *that is,*

$$\mathcal{A} = \mathcal{A}_{(u_1,u_2),\epsilon} = \{(a_1, a_2) \in [u_1, 1/2] \times [u_2, 1] \,:\, (81) \text{ holds}\},$$

which we show in Figure 6.

Proof of Lemma 5.4. Combine the restrictions in (70), (71), (78) and (79). □

From Lemma 5.4 we get one more restriction for (u_1, u_2). Indeed, (u_1, u_2) must satisfy (81) for every $\epsilon > 0$, so from $a_1 \le 1/2$, $a_2 \le 1$ and $ku_2 - u_1 \ge k - 1$ in (65), we find out that

$$\boxed{\frac{k-2}{k-1}\,u_1 + \frac{n(k-1)+1}{k-1}\,u_2 \le n + \frac{1}{2}.}$$

This last restriction, together with (65) and (69), as well as the basic restrictions (62), (63) and (64), determines the region of validity for (u_1, u_2), which we gather in the following lemma.

Lemma 5.6. *The parameters* (u_1, u_2) *must satisfy the conditions*

$$0 \le u_1 \le \frac{1}{2}, \qquad \frac{1}{2} \le u_2 \le 1,$$

as well as

$$ku_2 - u_1 \ge k - 1, \qquad \textit{condition } Q \ge 1, \tag{82}$$

$$u_2 - u_1 < 1 - \frac{1}{k}, \qquad \textit{condition for shrinking unit cell,} \tag{83}$$

and

$$\frac{k-2}{k-1}\,u_1 + \frac{n(k-1)+1}{k-1}\,u_2 \le n + \frac{1}{2}, \qquad \textit{condition for disjointness.} \tag{84}$$

The domain for (u_1, u_2) delimited in Lemma 5.6 plays a fundamental role. In particular, we want to emphasize the three conditions: $Q \ge 1$ (82), *shrinking unit cell* (83), and *disjointness* (84). To simplify references, we denote this domain by

$$\mathcal{D} = \{(u_1, u_2) \in [0, 1/2] \times [1/2, 1] \text{ subject to } (82), (83) \text{ and } (84)\},$$

which we show in Figure 7. Observe that the domain $\mathcal{D} = \mathcal{D}_{k,n}$ depends on both k and n.

Remark 5.7. *From Lemma 5.6 we can also deduce a lower bound for the condition for disjointness* (84). *For that, we find the line parallel to* (84) *that crosses the extreme point* $(u_1, u_2) = (0, 1 - 1/k)$, *which is the intersection of the conditions* $Q \ge 1$ (82) *and* $u_1 = 0$. *This line is*

$$\frac{k-2}{k-1}\,u_1 + \frac{n(k-1)+1}{k-1}\,u_2 = n - \frac{n-1}{k},$$

which implies

$$\boxed{n - \frac{n-1}{k} \le \frac{k-2}{k-1}\,u_1 + \frac{n(k-1)+1}{k-1}\,u_2 \le n + \frac{1}{2}.} \tag{85}$$

For every $(u_1, u_2) \in \mathcal{D}$ we need to prove that $\mathcal{A}_{(u_1,u_2),\epsilon} \neq \emptyset$, that is, that we can always pick a dilation $(a_1, a_2) \in \mathcal{A}_{(u_1,u_2),\epsilon}$.

Lemma 5.8. *If* $(u_1, u_2) \in \mathcal{D}$ *and* $\epsilon > 0$, *then* $\mathcal{A}_{(u_1,u_2),\epsilon} \neq \emptyset$.

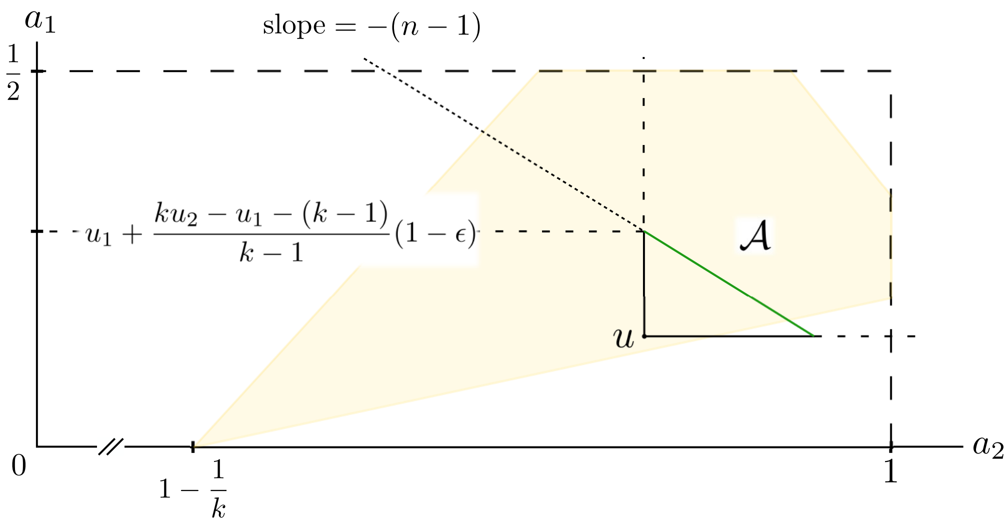


FIGURE 6. For every fixed $(u_1, u_2) \in \mathcal{D}$ (see Figure 7) the domain $\mathcal{A}$ for the exponent $\boldsymbol{a} = (a_1, a_2)$ is the solid green line.

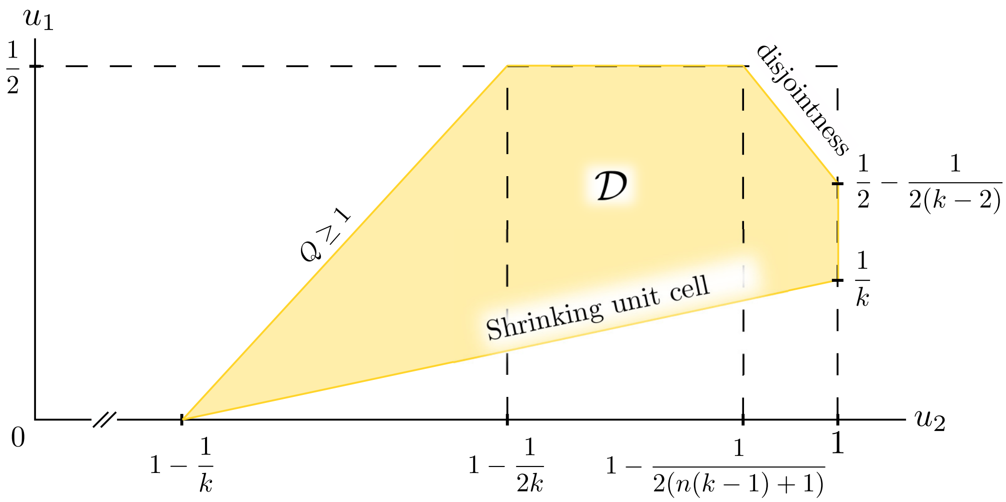


FIGURE 7. The domain $\mathcal{D}$ for the parameters (u_1, u_2). We represent the case $k \geq 5$. For $k = 2, 3$ and 4 the boundaries on the right are slightly different.

Proof. Fix $(u_1, u_2) \in \mathcal{D}$. We interpret (81) as a line in the variables (a_1, a_2). First, this line crosses the point

$$a_2 = u_2, \qquad a_1 = u_1 + \frac{ku_2 - u_1 - (k-1)}{k-1}(1-\epsilon) \geq u_1, \tag{86}$$

where the last inequality holds because $Q \geq 1$ (82), *i.e.* $ku_2 - u_1 \geq k - 1$. Since the line (81) has a negative slope equal to $-(n-1)$, the line crosses $[u_1, \infty] \times [u_2, \infty]$ (see Figure 6).

On the other hand, the condition for *disjointness* (84) implies that the line (81) satisfies

$$a_1 + (n-1)a_2 + \epsilon\Big(\frac{ku_2 - u_1}{k-1} - 1\Big) \leq n - \frac{1}{2},$$

That means that

$$a_2 = 1 \qquad \Longrightarrow \qquad a_1 \leq \frac{1}{2} - \epsilon\Big(\frac{ku_2 - u_1}{k-1} - 1\Big) \leq \frac{1}{2},$$

so there is an exponent $(a_1, a_2) \in [-\infty, 1/2] \times [-\infty, 1]$.

Summing up, the line (81) crosses both $[u_1, \infty] \times [u_2, \infty]$ and $[-\infty, 1/2] \times [-\infty, 1]$, so in particular it crosses $[u_1, 1/2] \times [u_2, 1]$. Thus, $\mathcal{A} \neq \emptyset$. $\square$

Remark 5.9. *Observe that the point in (86) depends exclusively on how far the point (u_1, u_2) is from the line $Q \geq 1$ (82).*

5.5. **A lower bound for the Hausdorff dimension.** We are ready to use the mass transference principle to prove a lower bound for the dimension of our set of divergence

$$F = \limsup_{m \to \infty} F_m = \bigcap_{J \in \mathbb{N}} \bigcup_{j > J} F_j, \tag{87}$$

where $F_m = F_{R_m}$ is defined in (58) with $R_m = 2^m$ for every $m \in \mathbb{N}$.

Proposition 5.10. *Let F be the set (87) with parameters $(u_1, u_2) \in \mathcal{D}$ as in Lemma 5.6 or Figure 7. If $\boldsymbol{a} = (a_1, a_2) \in \mathcal{A}$ as in Lemma 5.5 or Figure 6, then*

$$\dim_{\mathcal{H}} F \geq \min \{ a_1 + (n-1)a_2 + 1/2 , \ \ n - 1 + 2a_1 \} .$$

Consequently,

$$a_1 \geq (n-1)a_2 - (n - 3/2) \qquad \Longrightarrow \qquad \dim_{\mathcal{H}} F \geq a_1 + (n-1)a_2 + 1/2, \tag{88}$$

while

$$a_1 \leq (n-1)a_2 - (n - 3/2) \qquad \Longrightarrow \qquad \dim_{\mathcal{H}} F \geq n - 1 + 2a_1. \tag{89}$$

Proof. In Lemma 5.4 we showed that the dilated slabs $E_{p,q,R,\boldsymbol{a}}$ form a uniform local ubiquitous system, so we use Theorem 4.4 with $\boldsymbol{b} = (1/2, 1, \ldots, 1)$ and $\boldsymbol{a} = (a_1, a_2, \ldots, a_2)$. Thus,

$$\begin{aligned}\dim_{\mathcal{H}} F &= \dim_{\mathcal{H}} \limsup_{m \to \infty} F_m \\ &\geq \min_{A \in \{1, 1/2\}} \left\{ \sum_{\ell \in K_1(A)} 1 + \sum_{\ell \in K_2(A)} \left(1 - \frac{b_\ell - a_\ell}{A} \right) + \sum_{\ell \in K_3(A)} \frac{a_\ell}{A} \right\}.\end{aligned} \tag{90}$$

We compute that now:

- For $A = 1$ and $a_2 < 1$, we have

$$K_1 = \{\ell : a_\ell \geq 1\} = \emptyset, \quad K_2 = \{\ell : b_\ell \leq 1\} \setminus K_1 = \{1, \ldots, n\}, \quad K_3 = \{1, \ldots, n\} \setminus (K_1 \cup K_2) = \emptyset,$$

  so the number inside braces in (90) is

$$\alpha_1 := 1 - \frac{b_1 - a_1}{A} + (n-1) \left(1 - \frac{b_2 - a_2}{A} \right) = a_1 + (n-1)a_2 + 1/2.$$

  When $a_2 = 1$ the end result is the same.

- For $A = 1/2$ and $a_1 < 1/2$, first observe that Lemmas 5.4 and 5.6 imply $a_2 \geq u_2 \geq 1/2$, so

$$K_1 = \{\ell : a_\ell \geq 1/2\} = \{2, \ldots, n\}, \quad K_2 = \{\ell : b_\ell \leq 1/2\} \setminus K_1 = \{1\}, \quad K_3 = \{1, \ldots, n\} \setminus (K_1 \cup K_2) = \emptyset,$$

  so the number inside braces in (90) is

$$\alpha_2 := (n-1) + 1 - \frac{b_1 - a_1}{A} = n - \frac{1/2 - a_1}{1/2} = n - 1 + 2a_1.$$

  When $a_1 = 1/2$ the end result is the same.

Thus, $\dim_{\mathcal{H}} F \geq \min\{\alpha_1, \alpha_2\}$. Now, the boundary where the minimum changes is the line

$$a_1 = (n-1)a_2 - (n - 3/2), \tag{91}$$

from which we immediately get (88) and (89) in the statement. $\square$

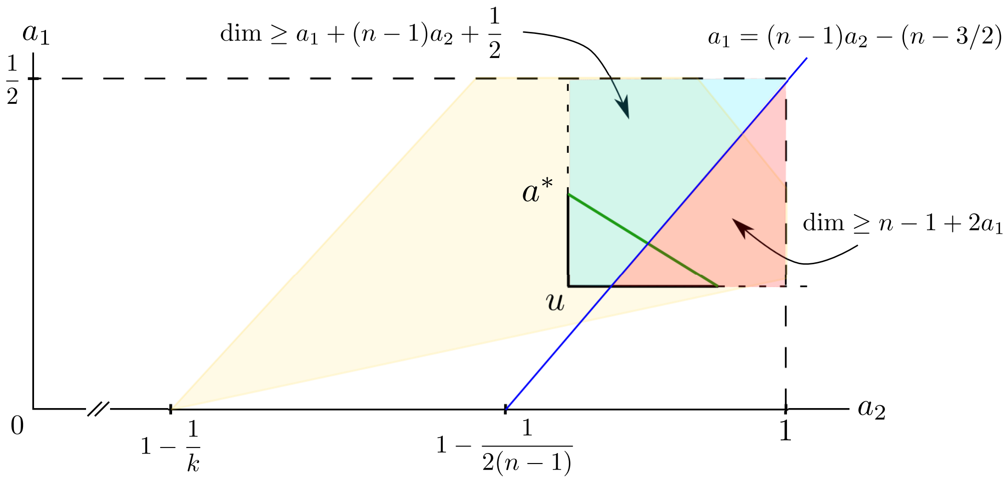


FIGURE 8. In blue, the boundary (91) that delimits the two different regions for $\boldsymbol{a}$ according to Proposition 5.10. In this case, the line $\mathcal{A}$ has parts in both sides of it.

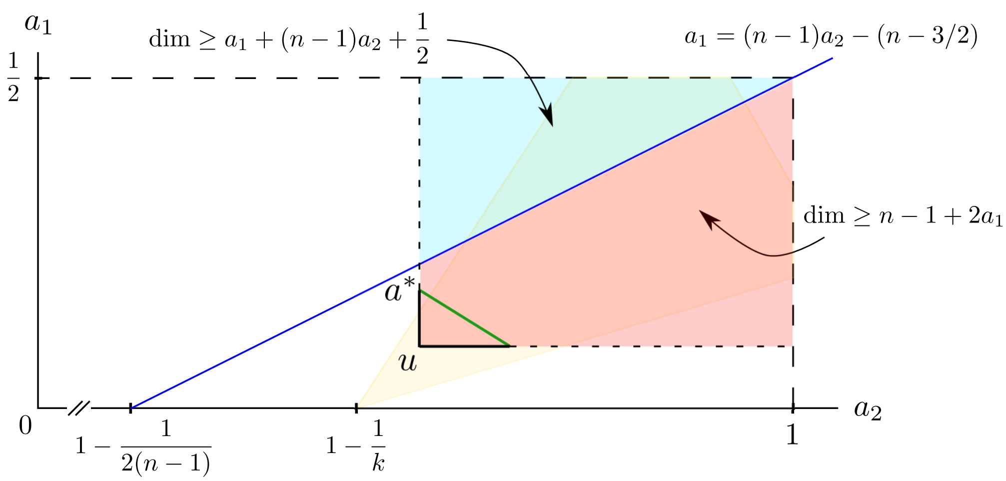


FIGURE 9. In blue, the boundary (91) that delimits the two different regions for $\boldsymbol{a}$ according to Proposition 5.10. In this case, the line $\mathcal{A}$ is completely below it.

Proposition 5.10 shows that different choices of $(a_1, a_2) \in \mathcal{A}$ give us different lower bounds for the dimension. Moreover, $\mathcal{A}$ depends on (u_1, u_2). Figure 8 shows that certain (u_1, u_2) allow $\mathcal{A}$ to intersect both sides of the boundary (91). However, for some others, Figure 9 suggests that $\mathcal{A}$ may lie completely below the boundary, so only the case (89) is possible. We now determine whether one case or the other happens.

Proposition 5.11. *Let $(u_1, u_2) \in \mathcal{D}$. Then,*

$$\left(n - 1 - \frac{k}{k-1}\right) u_2 - \frac{k-2}{k-1} u_1 < n - \frac{5}{2} \quad \Longrightarrow \quad \exists \epsilon, (a_1, a_2) \in \mathcal{A}_\epsilon \,:\, a_1 \geq (n-1)a_2 - (n-3/2).$$

On the contrary,

$$\left(n - 1 - \frac{k}{k-1}\right) u_2 - \frac{k-2}{k-1} u_1 \geq n - \frac{5}{2} \quad \Longrightarrow \quad a_1 < (n-1)a_2 - (n-3/2), \quad \forall \epsilon, (a_1, a_2) \in \mathcal{A}_\epsilon.$$

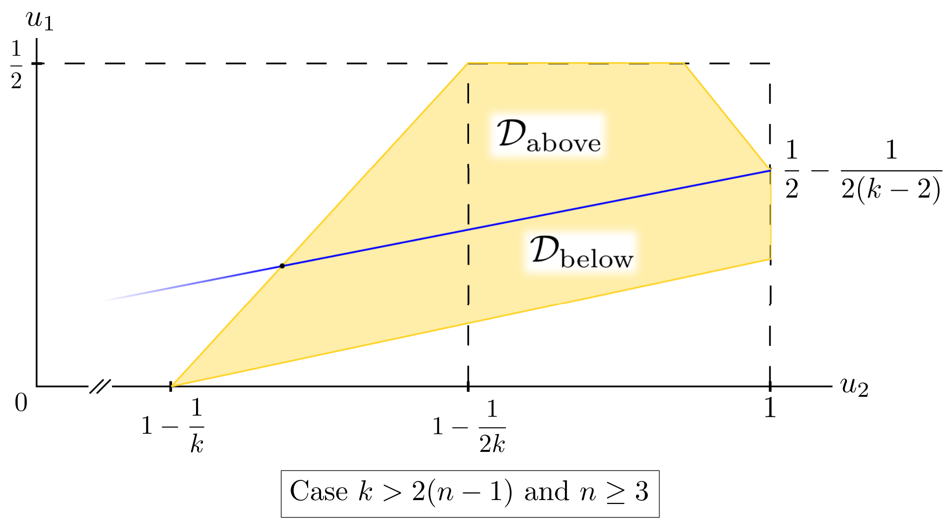


FIGURE 10. The separation of the domain $\mathcal{D}$ according to Proposition 5.11.

Remark 5.12. *This proposition determines two regions for* $(u_1, u_2) \in \mathcal{D}$*,*

$$\mathcal{D}_{above} = \left\{ (u_1, u_2) \in \mathcal{D} \mid \left(n - 1 - \frac{k}{k-1} \right) u_2 - \frac{k-2}{k-1} u_1 < n - \frac{5}{2} \right\}$$

and

$$\mathcal{D}_{below} = \left\{ (u_1, u_2) \in \mathcal{D} \mid \left(n - 1 - \frac{k}{k-1} \right) u_2 - \frac{k-2}{k-1} u_1 \geq n - \frac{5}{2} \right\},$$

which we show in Figure 10. They are important to calculate the Sobolev exponent in (41).

Proof of Proposition 5.11. We have to prove that

$$\left(n - 1 - \frac{k}{k-1} \right) u_2 - \frac{k-2}{k-1} u_1 \geq n - \frac{5}{2} \tag{92}$$

if and only if $a_1 < (n-1)a_2 - (n - 3/2)$ for every $\epsilon > 0$ and $(a_1, a_2) \in \mathcal{A}_\epsilon$, and by (81) the last inequality is the same as

$$2a_1 < \frac{k-2}{k-1} u_1 + \left(n + \frac{1}{k-1} \right) u_2 - n + \frac{1}{2} - \epsilon \left(\frac{ku_2 - u_1}{k-1} - 1 \right). \tag{93}$$

In Figures 8 and 9 we see that, for fixed $u \in \mathcal{D}$, the existence of at least one dilation $\boldsymbol{a}$ in the region (88) depends solely on the location of the point (a_1^*, a_2^*) where the lines $\mathcal{A}_\epsilon$ (81) and $a_2 = u_2$ intersect. In other words, it suffices to prove that (92) holds if and only if (93) holds for (a_1^*, a_2^*) and for every $\epsilon > 0$ because $a_1 \leq a_1^*$ for every $(a_1, a_2) \in \mathcal{A}$.

The point (a_1^*, a_2^*) is

$$a_2^* = u_2, \qquad a_1^* = \frac{k-2}{k-1} u_1 + \frac{k}{k-1} u_2 - 1 - \epsilon \left(\frac{ku_2 - u_1}{k-1} - 1 \right), \tag{94}$$

so (a_1^*, a_2^*) satisfies (93) for all $\epsilon > 0$ if and only if

$$\left(n - 1 - \frac{k}{k-1} \right) u_2 - \frac{k-2}{k-1} u_1 > n - 5/2 - \epsilon \left(\frac{ku_2 - u_1}{k-1} - 1 \right), \qquad \forall \epsilon > 0.$$

By the condition $Q \geq 1$ (82), this proves the proposition. $\square$

5.6. **The Hausdorff dimension of the divergence set.** Combining Propositions 5.10 and 5.11, we are now able to determine the Hausdorff dimension of the divergence set. Let us begin with an upper bound that comes from the natural covering of F.

Proposition 5.13. *Let $k \in \mathbb{N}$, $k \geq 2$. If F is the set defined in (87) with parameters $(u_1, u_2) \in \mathcal{D}$ as in Lemma 5.6, then*

$$\dim_{\mathcal{H}} F \leq \Big(n + \frac{1}{k-1}\Big)\, u_2 + \frac{k-2}{k-1}\, u_1 - \frac{1}{2}.$$

Proof. From the definition of the limsup, we have $F \subset \cup_{m \geq M} F_m$ for every $M \in \mathbb{N}$. We will first cover each F_m so that the union of all those coverings with $m \geq M$ cover F.

From its definition in (58), F_R is a union of slabs of side lengths $R^{-1/2}$ in the first coordinate and R^{-1} in the rest of coordinates. The smallest scale being R^{-1}, we cover each slab with approximately $R^{1/2}$ balls of radius R^{-1}. Since the number of slabs in $F_R \cap [-1,1]^n$ is approximately

$$\frac{D^k\, Q}{R^{k-1}}\, (D\, Q)^{n-1}\, Q = R^{u_1 + (n-1)\, u_2 + \frac{k u_2 - u_1}{k-1} - 1} = R^{(n+\frac{1}{k-1}) u_2 + \frac{k-2}{k-1} u_1 - 1},$$

we need around $R^{(n+\frac{1}{k-1})u_2 + \frac{k-2}{k-1}u_1 - \frac12}$ balls of radius R^{-1} to cover F_R. Now, the Hausdorff content of a set A is defined by

$$\mathcal{H}^s_\delta(A) = \inf \left\{ \sum_{j=1}^{\infty} (\operatorname{diam} U_j)^s \quad \Big| \quad A \subset \bigcup_{j=1}^{\infty} U_j \text{ such that } \operatorname{diam} U_j \leq \delta \right\}, \qquad \delta > 0. \tag{95}$$

Choosing $M \in \mathbb{N}$ such that $R_M^{-1} \leq \delta$, the union of the coverings above for $m \geq M$ gives

$$\mathcal{H}^s_\delta(F) \leq \sum_{m \geq M} R_m^{(n+\frac{1}{k-1})u_2 + \frac{k-2}{k-1}u_1 - \frac12}\, R_m^{-s}.$$

Taking $\delta \to 0$ implies $M \to \infty$, so if $s > (n + \frac{1}{k-1})u_2 + \frac{k-2}{k-1}u_1 - \frac12$, we get

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F) = 0.$$

This implies that $\dim_{\mathcal{H}} F \leq (n + \frac{1}{k-1})u_2 + \frac{k-2}{k-1}u_1 - \frac12$. ∎

This upper bound is valid for every $(u_1, u_2) \in \mathcal{D}$, but it is optimal only in one of the situations in Proposition 5.11.

Proposition 5.14. *Let $k \in \mathbb{N}$, $k \geq 2$. If F is the set (87) with parameters $(u_1, u_2) \in \mathcal{D}$ as in Lemma 5.6, then*

$$\Big(n - 1 - \frac{k}{k-1}\Big)\, u_2 - \frac{k-2}{k-1}\, u_1 < n - \frac52 \quad \Longrightarrow \quad \dim_{\mathcal{H}} F = \frac{k-2}{k-1}\, u_1 + \frac{n(k-1)+1}{k-1}\, u_2 - \frac12.$$

Proof. Proposition 5.11 ensures the existence of $(a_1, a_2) \in \mathcal{A}$ satisfying $a_1 \geq (n-1)a_2 - (n - 3/2)$, so according to Proposition 5.10,

$$\dim_{\mathcal{H}} F \geq a_1 + (n-1)a_2 + 1/2.$$

Moreover, from (81) in the definition of $\mathcal{A}$, we have

$$a_1 + (n-1)a_2 + \frac12 = \frac{k-2}{k-1}\, u_1 + \frac{n(k-1)+1}{k-1}\, u_2 - \frac12 - \epsilon\, \frac{k u_2 - u_1 - (k-1)}{k-1},$$

so we directly get

$$\dim_{\mathcal{H}} F \geq \frac{k-2}{k-1}\, u_1 + \frac{n(k-1)+1}{k-1}\, u_2 - \frac12 - \epsilon\, \frac{k u_2 - u_1 - (k-1)}{k-1}.$$

Since this holds for any $\epsilon > 0$ and by (82) we know that $ku_2 - u_1 \geq k-1$, we get

$$\dim_{\mathcal{H}} F \geq \frac{k-2}{k-1}\, u_1 + \frac{n(k-1)+1}{k-1}\, u_2 - \frac{1}{2}.$$

This lower bound matches the upper bound in Proposition 5.13, so the proof is complete. $\square$

We now tackle the complementary case.

Proposition 5.15. *Let $k \in \mathbb{N}$, $k \geq 2$. Let F be the set (87) with parameters $(u_1, u_2) \in \mathcal{D}$ as in Lemma 5.6. Then,*

$$\left(n - 1 - \frac{k}{k-1}\right) u_2 - \frac{k-2}{k-1}\, u_1 \geq n - \frac{5}{2} \quad \Longrightarrow \quad \dim_{\mathcal{H}} F = n - 3 + 2\,\frac{ku_2 + (k-2)u_1}{k-1}.$$

Proof. We prove the lower bound first. Proposition 5.11 implies that all exponents $(a_1, a_2) \in \mathcal{A}$ satisfy $a_1 < (n-1)a_2 - (n-3/2)$, so according to Proposition 5.10 we have

$$\dim_{\mathcal{H}} F \geq n - 1 + 2a_1, \qquad \forall (a_1, a_2) \in \mathcal{A}.$$

Let us choose the one with the largest a_1, which is always (94). Thus,

$$\dim_{\mathcal{H}} F \geq n - 3 + 2\frac{(k-2)u_1 + ku_2}{k-1} - 2\epsilon \frac{ku_2 - u_1 - (k-1)}{k-1}.$$

Since this holds for every $\epsilon > 0$ and by (82) we have $ku_2 - u_1 \geq k-1$, we get

$$\dim_{\mathcal{H}} F \geq n - 3 + 2\frac{(k-2)u_1 + ku_2}{k-1}.$$

Regarding the upper bound, the hypothesis implies that

$$n - 3 + 2\,\frac{ku_2 + (k-2)u_1}{k-1} \leq \frac{k-2}{k-1}\, u_1 + \frac{n(k-1)+1}{k-1}\, u_2 - \frac{1}{2},$$

so there is a gap with the upper bound in Proposition 5.13. Actually, in this case the covering used in Proposition 5.13 is not optimal and can be improved as follows:

From its definition in (58), let us rewrite F_R as

$$F_R = \bigcup_{Q/2 \leq q \leq Q} \bigcup_{p_1} \left[\bigcup_{p' : p \in G(q)} B_1\left(k\,\frac{R^{k-1}}{D^k}\frac{p_1}{q}, \frac{1}{R^{1/2}}\right) \times B_{n-1}\left(\frac{1}{D}\,\frac{p'}{q}, \frac{1}{R}\right)\right]. \tag{96}$$

Recall that we restrict to $[-1,1]^n$. Then, for every fixed q and p_1, we can cover the set inside the brackets using $R^{(n-1)/2}$ balls of radius $R^{-1/2}$. Since we have $\simeq Q$ choices for q and $\simeq QD^k/R^{k-1}$ choices for p_1, the number of balls of radius $R^{-1/2}$ that we need to cover F_R is

$$\frac{Q^2 D^k}{R^{k-1}}\, R^{(n-1)/2} = R^{u_1 + \frac{ku_2 - u_1}{k-1} - 1}\, R^{\frac{n-1}{2}} = R^{\frac{n-3}{2} + \frac{k-2}{k-1}u_1 + \frac{k}{k-1}u_2},$$

where we used (59) and (61). Thus, as in the proof of Proposition 5.13, taking $M \in \mathbb{N}$ such that $R_M^{-1} \leq \delta$, we get

$$\mathcal{H}^s_\delta(F) \leq \sum_{m \geq M} R_m^{\frac{n-3}{2} + \frac{k-2}{k-1}u_1 + \frac{k}{k-1}u_2}\, R_m^{-s/2}.$$

Thus,

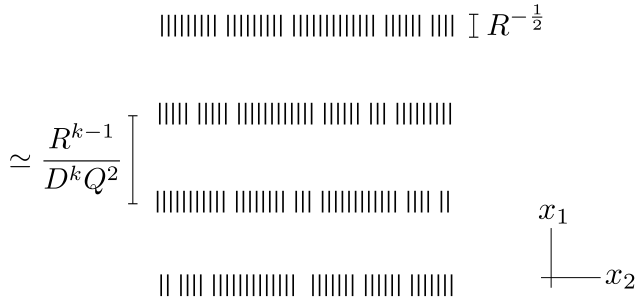
$$s > n - 3 + 2\left(\frac{k-2}{k-1}u_1 + \frac{k}{k-1}u_2\right) \quad \Longrightarrow \quad \mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F) = 0,$$

and consequently,

$$\dim_{\mathcal{H}} F \leq n - 3 + 2\frac{(k-2)u_1 + ku_2}{k-1}.$$

The proof is complete. $\square$

Remark 5.16. *In* (96)*, we are piling the slabs of* F_R *in sheets of width* $R^{-1/2}$ *in the direction* x_1*:*



These sheets are centered at $x_1 = k(R^{k-1}/D^k)(p_1/q)$*, so the distance between the centers of two sheets is*

$$k\frac{R^{k-1}}{D^k}\left(\frac{p_1}{q} - \frac{p_1'}{q'}\right) \geq k\frac{R^{k-1}}{D^k}\frac{1}{qq'} \simeq \frac{R^{k-1}}{D^k\,Q^2}.$$

Then, the sheets will be pairwise disjoint if this separation is greater than their width $R^{-1/2}$*. Using* (59) *and* (61)*,*

$$\frac{R^{k-1}}{D^k\,Q^2} > \frac{1}{R^{1/2}} \quad \iff \quad R^{-\frac{k}{k-1}\,u_2 - \frac{k-2}{k-1}\,u_1 + 1} > \frac{1}{R^{1/2}} \quad \iff \quad \frac{k-2}{k-1}\,u_1 + \frac{k}{k-1}\,u_2 \leq \frac{3}{2}.$$

And indeed, the hypothesis of Proposition 5.15 together with $u_2 \leq 1$ *implies precisely*

$$\frac{k-2}{k-1}\,u_1 + \frac{k}{k-1}\,u_2 \leq (n-1)u_2 - (n-5/2) \leq \frac{3}{2}.$$

5.7. **Some simpler cases.** We have fully determined the Hausdorff dimension of F in Propositions 5.14 and 5.15. For some k and n, those expressions can be simplified

5.7.1. *The dimension when* $k = 2$*.* This corresponds to the Schrödinger equation, which was already studied in [7, 29]. We include it here for the sake of completeness and of comparison with $k \geq 3$.

In this case, only Proposition 5.14 applies. Indeed, the hypothesis there turns into

$$(n-3)\,u_2 < n - 5/2,$$

which is always satisfied because:

- when $n = 2$, the condition is $u_2 \geq 1/2$, which is always satisfied as we saw in (66).
- when $n = 3$, the condition is $0 < 1/2$, which is trivially satisfied.
- when $n \geq 4$, we need

$$u_2 < \frac{n-5/2}{n-3} = \frac{n-3+1/2}{n-3} = 1 + \frac{1}{2(n-3)},$$

  which is also satisfied because we always have $u_2 \leq 1$.

Thus, Proposition 5.14 directly gives the following result.

Proposition 5.17. *Let* $k = 2$ *and* $(u_1, u_2) \in \mathcal{D}$*. Then,*

$$\dim_{\mathcal{H}} F = (n+1)\,u_2 - 1/2.$$

Remark 5.18. *The reader might want to compare this result with Theorems 12 and 14 in* [29]*. In the notation of that paper,* $u_1 = b$ *and* $u_2 = ((n-1)a + b + 1)/(n+1)$*, so we have not only computed the dimension over the "extremal" lines* $Q = 1$ *and* $u_1 = 1/2$ *as in* [29]*, but on the whole region* $\mathcal{D}$*.*

5.7.2. *The dimension when* $k = 3, 4$ *and* $n \geq 3$. Like for $k = 2$, only Proposition 5.14 applies because its hypothesis always holds, that is,

$$\left(n - 1 - \frac{k}{k-1}\right) u_2 - \frac{k-2}{k-1} u_1 < n - \frac{5}{2}. \tag{97}$$

Indeed, by the *shrinking unit cell* condition (83) we have $-u_1 < -u_2 + 1 - 1/k$, so

$$\left(n - 1 - \frac{k}{k-1}\right) u_2 - \frac{k-2}{k-1} u_1 < (n-3)u_2 + \frac{k-2}{k} \leq n - \frac{5}{2} + \frac{k-4}{2k},$$

where we used $u_2 \leq 1$ and $n \geq 3$ in the last inequality. Hence, (97) is always satisfied as long as $k \leq 4$, and Proposition 5.14 directly implies the following result.

Proposition 5.19. *Let* $k = 3, 4$ *and* $n \geq 3$*. If* $(u_1, u_2) \in \mathcal{D}$*, then*

$$\dim_{\mathcal{H}} F = \frac{k-2}{k-1} u_1 + \frac{n(k-1)+1}{k-1} u_2 - \frac{1}{2}.$$

6. Sobolev exponents for divergence

Once we know the dimension of F, let $0 < \alpha < n$ and fix $\dim_{\mathcal{H}} F = \alpha$. Recall that in (42) we built a counterexample that

- diverges in F,
- is in $H^s(\mathbb{R}^n)$ for every $s < s(\alpha)$ defined in (41), which by (59) and (60) is rewritten in terms of (u_1, u_2) as

$$s(u_2) = \frac{1}{4} + \frac{n-1}{2}(1 - u_2). \tag{98}$$

Our objective is to maximize this Sobolev exponent for every fixed α. For that, it suffices to minimize u_2 subject to the condition $\dim_{\mathcal{H}} F = \alpha$. Propositions 5.14 and 5.15 give the relationship between α and (u_1, u_2).

We first briefly solve the case $k = 2$ for the Schrödinger equation, and then we tackle the general case $k \geq 3$.

6.1. $\boldsymbol{k = 2}$ (Schrödinger equation).

In this case, by Proposition 5.17 we are fixing

$$\alpha = (n+1)\, u_2 - 1/2,$$

so the Sobolev exponent $s(u_2)$ in (98) is fully determined by

$$\begin{aligned} s(\alpha) = \frac{1}{4} + \frac{n-1}{2}\left(1 - \frac{\alpha + 1/2}{n+1}\right) &= \frac{1}{4} + \frac{n-1}{4(n+1)} + \frac{n-1}{2(n+1)}(n - \alpha) \\ &= \frac{n}{2(n+1)} + \frac{n-1}{2(n+1)}(n-\alpha). \end{aligned}$$

The conditions for $Q \geq 1$ (82) and for *disjointness* (84) imply that

$$1/2 \leq u_2 \leq \frac{n + 1/2}{n+1},$$

which restricts the choice of α to

$$n/2 \leq \alpha \leq n.$$

This result was obtained in [29].

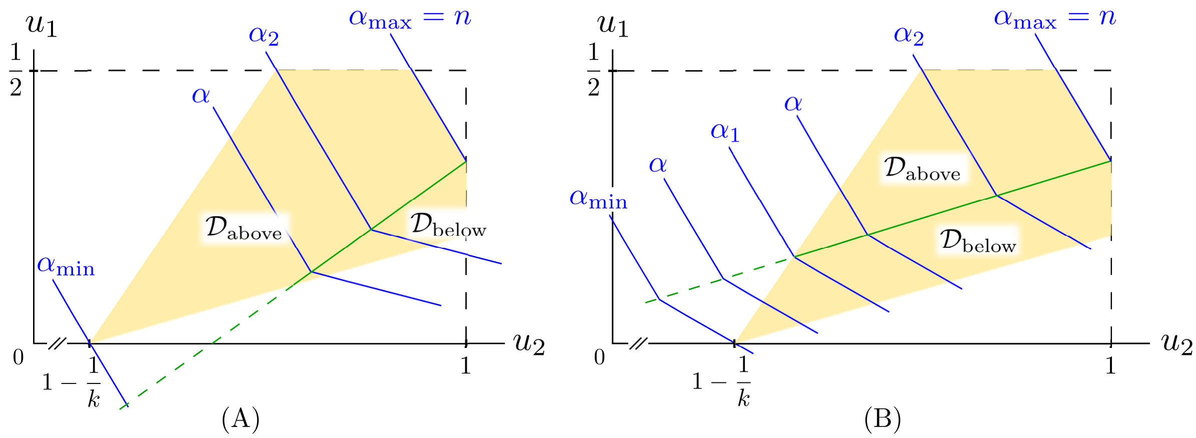


FIGURE 11. Case $n \geq 3$. The green line is the boundary $\mathcal{D}_{\text{above}}/\mathcal{D}_{\text{below}}$ (99).

6.2. $\boldsymbol{k \geq 3}$. In general, the region $\mathcal{D}$ for (u_1, u_2) is split into $\mathcal{D}_{\text{above}}$ and $\mathcal{D}_{\text{below}}$, defined in Remark 5.12 and with boundary in the line

$$\left(n - 1 - \frac{k}{k-1}\right) u_2 - \frac{k-2}{k-1} u_1 = n - \frac{5}{2}. \tag{99}$$

The dimension of the divergence set is given by Proposition 5.14 if $(u_1, u_2) \in \mathcal{D}_{\text{above}}$ and by Proposition 5.15 if $(u_1, u_2) \in \mathcal{D}_{\text{below}}$. Thus, for fixed α, we need (u_1, u_2) in the polygonal line

$$\alpha = \begin{cases} \dfrac{k-2}{k-1} u_1 + \dfrac{n(k-1)+1}{k-1} u_2 - \dfrac{1}{2}, & \text{for } (u_1, u_2) \in \mathcal{D}_{\text{above}}, \\ n - 3 + 2\,\dfrac{k u_2 + (k-2)u_1}{k-1}, & \text{for } (u_1, u_2) \in \mathcal{D}_{\text{below}}, \end{cases} \tag{100}$$

with smallest u_2. The broken line (100) is shown in Figure 11. If we see it as u_1 in function of u_2, it has negative slope and runs parallel to the line given by the condition for *disjointness* (84) in the region $\mathcal{D}_{\text{above}}$. That means that

$$\boxed{\text{smallest } u_2 \quad \equiv \quad \text{intersection of (100) with either } u_1 = 1/2 \text{ or } Q = 1 \text{ (82)}\ .} \tag{101}$$

The situation much depends on whether the boundary line (99) intersects $Q = 1$ (82) in the region $\mathcal{D}$ (like in Figure 11 B) or not (like in Figure 11 A). Observe that (99) crosses the point

$$u_1 = \frac{1}{2} - \frac{1}{2(k-2)}, \qquad u_2 = 1$$

and has slope

$$\text{slope of (99)} = \frac{(n-1)(k-1)-k}{k-2}.$$

Given that $k \geq 3$, this slope is positive if and only if $n \geq 3$.

- If $n = 2$, the slope of (99) is negative, and it always intersects $Q = 1$ (82) at the point
$$u_1 = \frac{1}{2} - \frac{1}{2(k-1)}, \qquad u_2 = \frac{k - 3/2}{k-1} = 1 - \frac{1}{2(k-1)}.$$
Since $0 \leq u_1 < 1/2$, we have $(u_1, u_2) \in \mathcal{D}$.
- If $n \geq 3$, the point in (99) with $u_1 = 0$ is
$$u_1 = 0, \qquad u_2 = \frac{(n-5/2)(k-1)}{(n-1)(k-1)-k}.$$

Given that $Q = 1$ (82) crosses the point $(0, \frac{k-1}{k})$, the intersection happens inside $\mathcal{D}$ (like in Figure 11 B) if and only if

$$\frac{(n-5/2)(k-1)}{(n-1)(k-1)-k} < \frac{k-1}{k} \quad \Longleftrightarrow \quad n-1 < \frac{k}{2}. \tag{102}$$

In this case, the crossing point is

$$u_1 = \frac{1}{2} - \frac{n-1}{2(k-(n-1))}, \qquad u_2 = 1 - \frac{1}{2(k-(n-1))}. \tag{103}$$

Observe that both the condition (102) and the crossing point (103) also work for $n = 2$.

Thus, according to (102), we have two different situations to treat.

6.2.1. *When $n-1 \geq k/2$.* In this case, the boundary $\mathcal{D}_{\text{above}}/\mathcal{D}_{\text{below}}$ (99) does not intersect the line $Q = 1$ (82), as shown in Figure 11 A. In particular, both $Q = 1$ (82) and $u_1 = 1/2$ are in $\mathcal{D}_{\text{above}}$. By (101), it suffices to consider the section of the broken line (100) in $\mathcal{D}_{\text{above}}$,

$$\alpha = \frac{k-2}{k-1}u_1 + \frac{n(k-1)+1}{k-1}u_2 - \frac{1}{2}. \tag{104}$$

The additional restriction (85) in Remark 5.7 delimits the range of α to

$$n - \frac{n-1}{k} \leq \alpha + \frac{1}{2} \leq n + \frac{1}{2} \quad \Longrightarrow \quad n - \frac{n-1}{k} - \frac{1}{2} \leq \alpha \leq n.$$

We have two cases:

- If the line (104) intersects $u_1 = 1/2$, then that intersection point has

  $$u_2 = \left(\alpha + \frac{1}{2(k-1)}\right)\frac{k-1}{n(k-1)+1},$$

  and from (98) we obtain the Sobolev exponent

  $$s(\alpha) = \frac{nk}{4(n(k-1)+1)} + \frac{(n-1)(k-1)}{2(n(k-1)+1)}(n-\alpha). \tag{105}$$

  It is important to notice that, in this case, α is restricted by $u_2 \geq (k-1/2)/k$, or equivalently,

  $$\alpha \geq \frac{n(k-1)+1}{k-1} \cdot \frac{k-1/2}{k} - \frac{1}{2(k-1)} = n - \frac{n-1}{2k} = \alpha_2.$$

- If the line (104) intersects $Q = 1$ (82), then the point of intersection is

  $$u_1 = \frac{n-1-k(n-\alpha-1/2)}{n+k-1}, \qquad u_2 = 1 - \frac{n-\alpha+1/2}{n+k-1},$$

  so from (98) we get the Sobolev exponent

$$s(\alpha) = \frac{2(n-1)+k}{4(n+k-1)} + \frac{n-1}{2(n+k-1)}(n-\alpha), \qquad \text{for } n - \frac{n-1}{k} - \frac{1}{2} \leq \alpha \leq n - \frac{n-1}{2k}. \tag{106}$$

6.2.2. *When $n-1<k/2$.* In this case, the boundary $\mathcal{D}_{\text{above}}/\mathcal{D}_{\text{below}}$ (99) intersects $Q=1$ (82) at the point (103), as shown in Figure 11 B. Thus, contrary to the previous case, there are values of α for which the broken line (100) crosses $Q=1$ (82) in the region $\mathcal{D}_{\text{below}}$. The largest of such α, call it α_1, corresponds to the broken line (100) crossing the point (103). Since the broken line takes in $\mathcal{D}_{\text{below}}$ the form

$$\alpha = n-3+2\,\frac{ku_2+(k-2)u_1}{k-1}, \tag{107}$$

we get

$$\alpha_1 = n - \frac{n-1}{k-(n-1)}.$$

On the other hand, the smallest α, call it $\alpha_{\min}$, corresponds to the broken line (100) or (107) crossing the point $(u_1,u_2)=(0,(k-1)/k)$, that is,

$$\alpha_{\min} = n-1.$$

Thus, for $\alpha_{\min}\le\alpha\le\alpha_1$, the point of intersection between the broken line (107) and $Q=1$ (82) is

$$u_1 = \frac{1}{2}-\frac{n-\alpha}{2}, \qquad u_2 = 1-\frac{n-\alpha+1}{2k},$$

so from (98) we obtain the Sobolev exponent

$$s(\alpha) = \frac{1}{4}+\frac{n-1}{4k}+\frac{n-1}{4k}(n-\alpha).$$

Observe that this counterexample is only relevant if $s(\alpha)\ge(n-\alpha)/2$. Thus, we need to shrink the range of α so that

$$s(\alpha) = \frac{1}{4}+\frac{n-1}{4k}+\frac{n-1}{4k}(n-\alpha), \qquad \text{for} \quad n-\frac{1}{2}-\frac{3(n-1)}{2(2k-n+1)}\le\alpha\le n-\frac{n-1}{k-(n-1)}.$$

For the remaining $\alpha > n-\frac{n-1}{k-(n-1)}=\alpha_1$, the broken line (100) intersects either $Q=1$ (82) or $u_1=1/2$ in the region $\mathcal{D}_{\text{above}}$ in the same way as it did in the previous case $n-1\ge k/2$. Therefore, we obtain the same Sobolev exponents (105) and (106).

This completes the proof of Theorem 1.4.

7. Divergence for the saddle-like symbol

We now prove Theorem 1.5. For that, we need to build counterexamples for the symbols

$$P(\xi) = \xi_1^2+\cdots+\xi_m^2-\xi_{m+1}^2-\cdots-\xi_n^2 \tag{108}$$

with index $1\le m\le n/2$. Although Theorem 1.4 can be used in this case as well, it does not yield the largest possible sets of divergence. Instead, we adjust the example of Rogers, Vargas and Vega [32] to the fractal context.

7.1. When $\boldsymbol{\alpha\ge n-m+1}$: Optimal result.

We first prove that if α is large enough, the non-dispersive threshold in Theorem 1.1 is optimal for the symbol (108). That means that the solution T_t may behave as if there were no dispersion.

Theorem 7.1. *Let P be a quadratic form with index $1\le m\le n/2$. Assume that $\alpha\ge n-m+1$ and that*

$$s<\frac{n-\alpha+1}{2}.$$

Then, there exists $f\in H^s(\mathbb{R}^n)$ such that T_tf diverges in a set of Hausdorff dimension α.

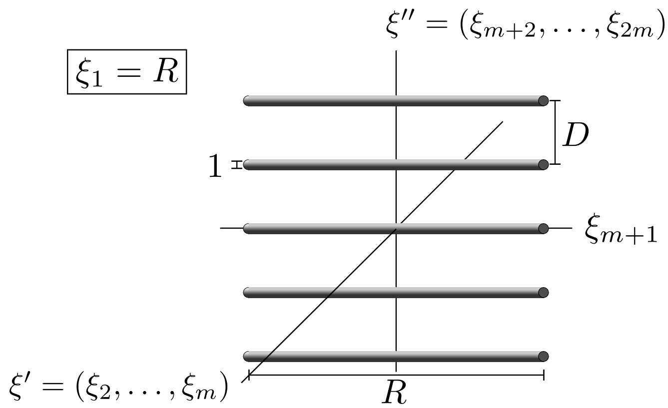
Proof. As usual, we build functions f_R at fixed scales $R \gg 1$ and then sum all of them after verifying that the solutions $T_t f_R$ do not interfere with each other.

First, the change of variables $\xi_i \to \frac{1}{2}(\xi_i + \xi_{m+i})$ and $\xi_{m+i} \to \frac{1}{2}(\xi_i - \xi_{m+i})$, for $i = 1, \ldots, m$, transforms the symbol to

$$P(\xi) = \xi_1\xi_{m+1} + \cdots + \xi_m\xi_{2m} - \xi_{2m+1}^2 - \cdots - \xi_n^2. \tag{109}$$

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\widehat{\varphi}$ supported in $B(0,c)$ for some small $c > 0$, and consider the preliminary initial datum

$$\widehat{f}_R(\xi) = \sum_{|l''| \le R/D} \widehat{\varphi}(\xi_1 - R, \xi', \xi_{m+1}/R, \xi'' - Dl'', \xi'''), \qquad l'' = (l_{m+2}, \ldots, l_{2m}) \in \mathbb{R}^{m-1},$$



Here, we are denoting $\xi = (\xi_1, \xi', \xi_{m+1}, \xi'', \xi''') \in \mathbb{R}^n$, where $\xi' = (\xi_2, \ldots, \xi_m)$, $\xi'' = (\xi_{m+2}, \ldots, \xi_{2m})$ and $\xi''' = (\xi_{2m+1}, \ldots, \xi_n)$. The solution can be written as

$$\begin{aligned} T_t f_R(x) &= \sum_{|l''| \le R/D} \int \widehat{\varphi}(\xi_1 - R, \xi', \xi_{m+1}/R, \xi'' - Dl'', \xi''')\, e(x \cdot \xi + tP(\xi))\, d\xi \\ &= R \sum_{|l''| \le R/D} e^{2\pi i (Rx_1 + x'' \cdot Dl'')} \\ &\qquad \int \widehat{\varphi}(\xi)\, e\Big(x_1\xi_1 + (x' + tDl'') \cdot \xi' + R(x_{m+1} + R\,t)\,\xi_{m+1} + \\ &\qquad\qquad + x'' \cdot \xi'' + x''' \cdot \xi''' + t(R\xi_1\xi_{m+1} + \xi' \cdot \xi'' - |\xi'''|^2)\Big)\, d\xi. \end{aligned}$$

We evaluate it at times $0 \le t \le 1/R$ and in the set F given by the conditions

$$|(x_1, x')| \le 1; \quad x_{m+1} + Rt = 0; \quad x'' = D^{-1}\mathbb{Z}^{m-1} + \mathcal{O}(R^{-1}); \quad \text{and} \quad |x'''| \le 1, \tag{110}$$

see Figure 12. Take the absolute value so that

$$|T_t f_R(x)| \simeq R \Big| \sum_{|l''| \le R/D} e^{2\pi i x'' \cdot Dl''} \Big| \simeq R \left(\frac{R}{D}\right)^{m-1}.$$

Since $\|f_R\|_2 \simeq R^{1/2}(R/D)^{\frac{m-1}{2}}$, we get

$$\frac{|T_t f_R(x)|}{\|f_R\|_2} \simeq R^{1/2} \left(\frac{R}{D}\right)^{\frac{m-1}{2}} = R^{\frac{1+(m-1)(1-a)}{2}},$$

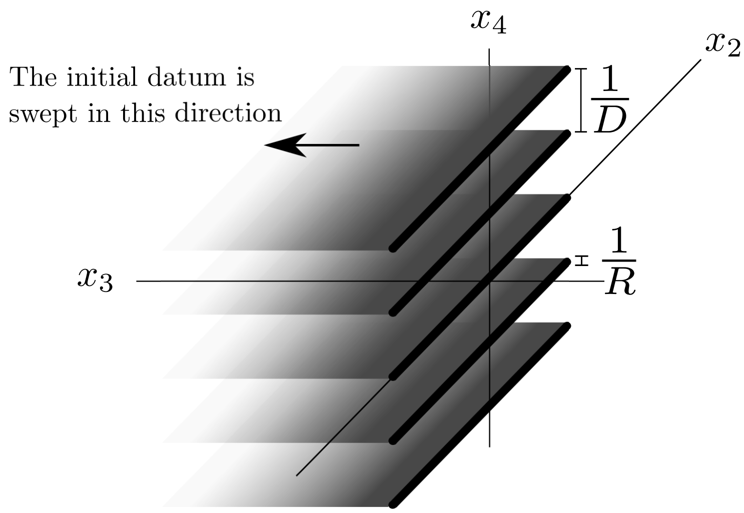


FIGURE 12. A representation of the example in Theorem 7.1 at a fixed scale R when $n = 4$ and $P(\xi) = \xi_1\xi_3 + \xi_2\xi_4$, *i.e.* $m = 2$. We can only represent a general slice $x_1 =$ constant. The initial datum f is concentrated around the dark rods in $x_3 = 0$ and, as time progresses, the rods are translated to the left until $x_3 = -1$. Thus, $\sup_{0\le t\le 1/R}|T_t f|$ is large along the sheets, which cover a set of dimension ≥ 2 for each $|x_1| \le 1$.

where we set $D = R^a$ with $0 \le a \le 1$. Denote

$$s(a) = \frac{1 + (m-1)(1-a)}{2}, \tag{111}$$

which is the exponent that will determine the regularity of the datum.

Define the scales $R_k = 2^k$ for $k \in \mathbb{N}$, and for some large enough $k_0 \in \mathbb{N}$ we define the datum

$$f(x) = \sum_{k\ge k_0} k \, \frac{f_{R_k}}{R_k^{s(a)} \, \|f_{R_k}\|_2}. \tag{112}$$

The triangle inequality gives $f \in H^s(\mathbb{R}^n)$ for all $s < s(a)$. Denoting by F_k the corresponding set in (110), the same method we used to prove Proposition 3.9 proves that for every $K \ge k_0$ and for every $x \in F_K$, there exists a time $t = t(x) \le 1/R_K$ such that

$$\left|T_{t(x)} f(x)\right| = \left| \sum_{k\ge k_0} k \, \frac{T_{t(x)} f_{R_k}(x)}{R_k^{s(a)} \, \|f_{R_k}\|_2} \right| \gtrsim K \, \frac{|T_{t(x)} f_{R_K}(x)|}{R_K^{s(a)} \, \|f_{R_K}\|_2} \simeq K.$$

Consequently, taking the limit $K \to \infty$, we get

$$\limsup_{t\to 0} |T_t f(x)| = \infty, \qquad \forall x \in F, \tag{113}$$

where $F = \limsup_{k\to\infty} F_k$.

It only remains to compute the Hausdorff dimension of F. Let us define the set $H = \limsup_{k\to\infty} H_k$, where

$$H_k = \left\{ x'' \in [-1,1]^{m-1} \, : \, |x'' - D_k^{-1} p''| \le c R_k^{-1} \text{ for some } p'' \in \mathbb{Z}^{m-1} \right\}, \qquad k \in \mathbb{N}.$$

For every $k \in \mathbb{N}$, the set F_k is essentially equal to $H_k \times [-1,1]^{n-m+1}$, where $[-1,1]^{n-m+1}$ is a cube in the variables (x_1, x', x_{m+1}, x'''). Then, $\dim_{\mathcal{H}} F = \dim_{\mathcal{H}} H + n - m + 1$; see [18, Corollary 7.4].

Since H is covered by $\bigcup_{k\geq K} H_k$ for all $K\in\mathbb{N}$, and H_k is a union of D_k^{m-1} balls of radius R_k^{-1}, we get

$$\mathcal{H}^s_{R_K^{-1}}(H) \leq \sum_{k\geq K} D_k^{m-1} R_k^{-s} = \sum_{k\geq K} R_k^{a(m-1)-s}, \qquad \forall K\in\mathbb{N},$$

so letting $K\to\infty$ we get $\mathcal{H}^s(H)=0$ as long as $s>a(m-1)$, which in turn implies that $\dim_{\mathcal{H}} H \leq a(m-1)$.

For the lower bound, we use the mass transference principle in its original form by Beresnevich and Velani in Theorem 4.1. For that, we need to find $s\geq 0$ such that $\limsup_{k\to\infty} H_k^s$ has full Lebesgue measure, where

$$H_k^s = \bigcup_{p''\in\mathbb{Z}^{m-1}} B\left(\frac{p''}{D_k}, \frac{1}{R_k^{s/(m-1)}}\right), \qquad k\in\mathbb{N}.$$

In particular, it suffices that each H_k^s fills the space, which happens when $R_k^{s/(m-1)}\simeq D_k$, that is, when $s=a(m-1)$. Thus, the mass transference principle implies $\dim_{\mathcal{H}} H\geq a(m-1)$. Consequently, $\dim_{\mathcal{H}} H = a(m-1)$ and

$$\dim_{\mathcal{H}} F = a(m-1)+n-(m-1) = n-(1-a)(m-1), \qquad \forall a\in[0,1]. \tag{114}$$

In particular, the range for the dimension is $[n-m+1,n]$.

To conclude the proof, let $\alpha\in[n-m+1,n]$ and fix $\dim_{\mathcal{H}} F=\alpha$. Then, according to (114), we need to choose a such that $n-\alpha=(1-a)(m-1)$, so the critical regularity in (111) is $s(\alpha)=(n-\alpha+1)/2$. Thus, by (113), we found a datum (112) that belongs to $H^s(\mathbb{R}^n)$ for every $s<(n-\alpha+1)/2$ and that diverges in a set F of Hausdorff dimension α. □

7.2. **When $\boldsymbol{\alpha < n-m+1}$.** In this case, the smoothing effect of T_t seems unavoidable. Combining the counterexample of Rogers, Vargas and Vega [32] with the Talbot effect in the counterexample of the proof of Theorem 1.4, we give a lower bound for $s_c(\alpha)$ that is strictly smaller than the non-dispersive threshold. The proof is similar to that of Theorem 1.4; we sketch it here and skip many details that can be found in Sections 3 and 5.

Theorem 7.2. *Let P be a quadratic form with index $1\leq m\leq n/2$. Assume that $n/2\leq\alpha<n-m+1$ and that*

$$s<\frac{n+(n-2m)(n-\alpha)}{2(n-2m+2)}. \tag{115}$$

Then, there exists $f\in H^s(\mathbb{R}^n)$ such that T_tf diverges in a set of Hausdorff dimension α.

Remark 7.3. *Let $s(\alpha)$ be right hand side of (115). Then $s(n-m+1)=m/2$ matches the exponent of Theorem 7.1. On the other hand, $s(n/2)=n/4$ coincides with the exponent in Corollary 1.3. We show this graphically in Figure 2.*

Proof. As in Theorem 7.1, it suffices to work with the symbol (109). Given $\varphi\in\mathcal{S}(\mathbb{R}^n)$ with $\widehat{\varphi}$ supported in $B(0,c)$ for some small $c>0$ and the parameter $R\gg 1$, we propose the datum

$$\widehat{f}_R(\xi) = \sum_{\substack{|l'''|\leq R/D\\ l'''\in\mathbb{Z}^{n-2m}}} \widehat{\varphi}(\xi_1-R,\xi',\xi_{m+1}/R,\xi''/R,\xi'''-Dl''').$$

Here, $D = R^\gamma$ for some $0 < \gamma < 1$, and we split the variable $\xi = (\xi_1, \xi', \xi_{m+1}, \xi'', \xi''')$. Moreover, $\|f_R\|_2 \simeq R^{m/2}(R/D)^{(n-2m)/2}$. The solution takes the form

$$\begin{aligned} T_t f_R(x) &= \sum_{|l'''|<R/D} \int \widehat{\varphi}(\xi_1 - R, \xi', \xi_{m+1}/R, \xi''/R, \xi''' - Dl''')\, e(x\cdot\xi + t\,P(\xi))\, d\xi \\ &= R^m\, e(Rx_1) \sum_{|l'''|<R/D} e(x'''\cdot Dl''' - tD^2|l'''|^2) \\ &\qquad \int \widehat{\varphi}(\xi)\, e\Big(x_1\xi_1 + x'\cdot\xi' + R(x_{m+1}+Rt)\xi_{m+1} + Rx''\cdot\xi'' + \xi'''\cdot(x''' - 2tDl''') \\ &\qquad\qquad + t\left(R\xi_1\xi_{m+1} + R\xi'\cdot\xi'' - |\xi'''|^2\right)\Big)\, d\xi. \end{aligned}$$

We impose $t < 1/R$, as well as

$$|x_1|, |x'|, |x'''| < 1, \quad x'' = 0 \quad \text{and} \qquad R\,|x_{m+1} + Rt| < 1, \tag{116}$$

so that the phase of the integral is small, and thus

$$|T_t f_R(x)| \simeq R^m \left| \sum_{|l'''|<R/D} e\left(x'''\cdot Dl''' - tD^2|l'''|^2\right)\right|.$$

Let $p''' \in \mathbb{Z}^{n-2m}$, $p_{m+1} \in \mathbb{Z}$, q a prime, and

$$x''' = \frac{1}{D}\frac{p'''}{q} \qquad \text{and} \qquad t = \frac{1}{D^2}\frac{p_{m+1}}{q}. \tag{117}$$

Assuming that $q \simeq Q$ and by periodicity, we transform the expression above into a Gauss sum,

$$|T_t f_R(x)| \simeq R^m \left(\frac{R}{DQ}\right)^{n-2m} \left| \sum_{l''' \in \mathbb{F}_q^{n-2m}} e\left(\frac{p'''\cdot l''' - p_{m+1}|l'''|^2}{q}\right)\right| \tag{118}$$

By Deligne's Theorem 3.1 and Lemma 3.6, we know that there is a set $G(q) \subset \mathbb{Z}^{n-2m+1}$ such that $|G(q)\cap[0,q]^{n-2m+1}| \simeq q^{n-2m+1}$ and for values $(p_{m+1}, p''') \in G(q)$ the Gauss sum in (118) is $\simeq q^{(n-2m)/2}$. As a consequence,

$$|T_t f_R(x)| \simeq R^m \left(\frac{R}{DQ^{1/2}}\right)^{n-2m} \qquad \Longrightarrow \qquad \frac{|T_t f_R(x)|}{\|f_R\|_2} \simeq R^{m/2}\left(\frac{R}{DQ}\right)^{(n-2m)/2}. \tag{119}$$

This result is analogue to Proposition 3.8 and will eventually give the desired Sobolev exponent. As in (112), the counterexample is built summing f_{R_k} for different scales $R_k = 2^k$, which diverges in the set $F = \limsup_{k\to\infty} F_k$, where $F_k = F_{R_k}$ are defined by the conditions (116) and (117). The procedure is the same as in Subsection 3.2, so we do not repeat it here.

Instead, let us focus on the Hausdorff dimension of F. Conditions (116) and (117) suggest defining

$$H_R = ([-1,0]\times[-1,1]^{n-2m}) \cap \bigcup_{q\simeq Q}\ \bigcup_{p\in G(q)} B\left(\frac{R}{D^2}\frac{p_{m+1}}{q}, \frac{1}{R}\right) \times B\left(\frac{1}{D}\frac{p'''}{q}, \frac{1}{R}\right).$$

Indeed, adding an error $|\epsilon| \leq 1/R$ to the choice of x''' does not alter the result of the Gauss sum. Calling $H_{R_k} = H_k$, let $H = \limsup_{k\to\infty} H_k$. Since the rest of the variables can be taken $(x_1, x') \in [-1,1]^m$ and $x'' = 0$, the final set $F \subset \mathbb{R}^n$ will have $\dim_{\mathcal{H}} F = \dim_{\mathcal{H}} H + m$.

To compute $\dim_{\mathcal{H}} H$, let us define the geometric parameters u_1, u_2 by

$$R^{u_1} = \frac{D^2 Q}{R}, \quad R^{u_2} = DQ \qquad \Longleftrightarrow \qquad Q = R^{2u_2 - u_1 - 1}, \quad D = R^{u_1 - u_2 + 1}.$$

With them, we easily get an upper bound for $\dim_{\mathcal{H}} H$. Indeed, H_R is covered by $QR^{u_1+(n-2m)u_2}$ balls of radius $1/R$. Since H is covered by $\bigcup_{k\geq K} H_k$, we immediately get

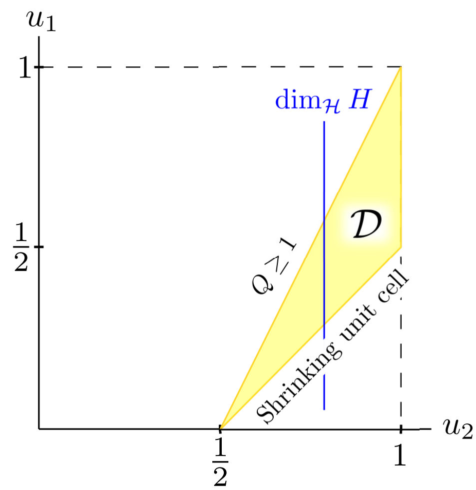
$$\mathcal{H}^s_{1/R_K}(H) \leq \sum_{k\geq K} QR^{u_1+u_2(n-2m)-s} = \sum_{k\geq K} R^{u_2(n-2m+2)-1-s},$$

and letting $K\to\infty$ implies that

$$\dim_{\mathcal{H}} H \leq u_2(n-2m+2)-1. \tag{120}$$

For the lower bound, we use the mass transference principle in Theorem 4.4. The original exponents are now $\boldsymbol{b}=(1,\dots,1)$, and the restrictions we have for u_1,u_2 are

- Basic separation restrictions: $0\leq u_1,u_2\leq 1$;
- $Q\geq 1$: $2u_2-u_1-1\geq 0$;
- Shrinking unit cell: $u_2-u_1\leq 1/2$.



Like we did in Subsection 5.3, to check that the slabs form a uniform local ubiquity system it is enough to find $\boldsymbol{a}=(a_1,a_2,\dots,a_2)$ such that $\mathcal{H}^n(\Omega_{R,\boldsymbol{a}})\geq c>0$, where

$$\Omega_{R,\boldsymbol{a}} = \bigcup_{q\simeq Q}\ \bigcup_{p\in G(q)\cap[0,q]^{n-2m+1}} B\left(\frac{p_{m+1}}{q},\frac{D^2}{R^{1+a_1}}\right)\times B\left(\frac{p'''}{q},\frac{D}{R^{a_2}}\right).$$

According to Proposition 5.2, given any $\epsilon>0$, it is enough to ask

$$\frac{D^2}{R^{1+a_1}}\left(\frac{D}{R^{a_2}}\right)^{n-2m} \simeq \frac{1}{Q^{n-2m+2-\epsilon}} \iff a_1+(n-2m)a_2 = u_1+(n-2m)u_2+(1-\epsilon)(2u_2-u_1-1), \tag{121}$$

where additionally $u_1\leq a_1<1$ and $u_2\leq a_2<1$. Given any valid u_1,u_2, there exist such a_1,a_2, so we apply the mass transference principle in Theorem 4.4 to get

$$\dim_{\mathcal{H}} H \geq \sum_{\ell\in K_1} 1 + \sum_{\ell\in K_2}(1-(b_\ell-a_\ell)) + \sum_{\ell\in K_3} a_\ell,$$

where $K_2=\{\ell : b_\ell\leq 1\}$ takes all the coordinates. In particular, $K_1=\emptyset=K_3$, so

$$\dim_{\mathcal{H}} H \geq a_1+a_2(n-2m).$$

Using (121), we rewrite it as

$$\dim_{\mathcal{H}} H \geq (n-2m+2)u_2-1-\epsilon(2u_2-u_1-1).$$

This is valid for every $\epsilon>0$, so given that $2u_2-u_1-1\geq 0$ and together with (120), we get

$$\dim_{\mathcal{H}} H = u_2(n-2m+2)-1.$$

Consequently, $\dim_{\mathcal{H}} F = u_2(n-2m+2)+m-1$.

To conclude, let us fix $\alpha = u_2(n-2m+2)+m-1$. Since $1/2 \le u_2 \le 1$, the dimension is restricted to $n/2 \le \alpha \le n-m+1$. The Sobolev exponent, which is the exponent of R in the last expression in (119), is $m/2+(1-u_2)(n-2m)/2$, so replacing u_2 for α we get

$$s(\alpha) = \frac{n+(n-2m)(n-\alpha)}{2(n-2m+2)}.$$

$\square$

7.3. **When n is odd, $m = (n-1)/2$ and $\alpha < (n+3)/2$.** In this case, the exponent in the preceding subsection can be improved.

Theorem 7.4. *Let n be odd and P be a quadratic form with index $m = (n-1)/2$. Assume that $(n+1)/2 \le \alpha \le (n+3)/2$ and that*

$$s < \frac{n-\alpha+m+1}{4}, \tag{122}$$

Then, there exists $f \in H^s(\mathbb{R}^n)$ such that $T_t f$ diverges in a set of Hausdorff dimension α.

Remark 7.5. *Let $s(\alpha)$ be right hand side of (122) and $s_T(\alpha)$ the regularity in (115). At one end point of α we have $s(n-m+1) = s_T(n-m+1)$, while the slopes of $s(\alpha)$ and $s_T(\alpha)$ are $-1/4$ and $-(n-2m)/[2(n-2m+2)]$, respectively. Hence, $s(\alpha) > s_T(\alpha)$ for $\alpha \le n-m+1$ if and only if $m > n/2-1$.*

Proof. The example is similar to the one in the proof of Theorem 7.1, and in particular, we assume the same presentation of P as

$$P(\xi) = \xi_1\xi_{m+1} + \cdots + \xi_m\xi_{2m} - \xi_n^2.$$

We consider the initial datum

$$\widehat{f_R}(\xi) = \sum_{|l| \le R^{1/2}/D} \widehat{\varphi}(\xi_1 - R, \xi', \xi_{m+1}/R, \xi''/R, \xi_n - lD),$$

where $(\xi_1, \dots, \xi_m, \xi_{m+1}, \dots, \xi_{2m}, \xi_n) = (\xi_1, \xi', \xi_{m+1}, \xi'', \xi_n)$. The solution can be written as

$$\begin{aligned} T_t f_R(x) &= \sum_{|l| \le R^{1/2}/D} \int \widehat{\varphi}(\xi_1 - R, \xi', \xi_{m+1}/R, \xi''/R, \xi_n - lD)\, e(x\cdot\xi + tP(\xi))\, d\xi \\ &= R^m \sum_{|l| \le R^{1/2}/D} e^{2\pi i (Rx_1 + x_n \cdot lD)} \\ &\qquad \int \widehat{\varphi}(\xi)\, e\Big(x_1\xi_1 + x'\cdot\xi' + R(x_{m+1}+Rt)\xi_{m+1} + Rx''\cdot\xi'' + \\ &\qquad\qquad (x_n - 2tlD)\cdot\xi_n + tR(\xi_1,\xi')\cdot(\xi_{m+1},\xi'') - t(\xi_n^2 + |lD|^2)\Big)\, d\xi. \end{aligned}$$

We evaluate it at times $0 \le t \le 1/R$ and in the set F given by the conditions

$$|(x_1, x')| \le 1; \quad x_{m+1} + Rt = 0; \quad x'' = 0; \quad \text{and} \quad x_n = D^{-1}\mathbb{Z} + \mathcal{O}(R^{-1/2}). \tag{123}$$

Take the absolute value so that

$$|T_t f_R(x)| \simeq R^m \Big| \sum_{|l| \le R^{1/2}/D} e^{2\pi i x_n \cdot lD} \Big| \simeq R^m \frac{R^{1/2}}{D}.$$

Since $\|f_R\|_2 \simeq R^{m/2}(R^{1/2}/D)^{1/2}$, we get

$$\frac{|T_t f_R(x)|}{\|f_R\|_2} \simeq R^{m/2}\left(\frac{R^{1/2}}{D}\right)^{1/2} = R^{(n-2a)/4},$$

where we set $D = R^a$ with $0 \le a \le 1/2$. Denote

$$s(a) = \frac{n-2a}{4}, \tag{124}$$

which is the exponent that will determine the regularity of the datum.

We define the scales $R_k = 2^k$ for $k \in \mathbb{N}$ and set the initial datum f as in (112), so $f \in H^s(\mathbb{R}^n)$, for $s < s(a)$, and $T_t f$ diverges in the set $F = \limsup_{k\to\infty} F_k$, where F_k are the sets given by (123). To compute the dimension of F, notice that it is a product $F_k = [-1,1]^{m+1} \times H_k$, where

$$H_k = \left\{ x_n \in [-1,1] \,:\, |x_n - D_k^{-1} p| \le c R_k^{-1/2} \text{ for some } p \in \mathbb{Z} \right\}, \qquad k \in \mathbb{N}.$$

By the same methods used in the previous subsections, the dimension of this set is $2a$, so $\alpha = \dim_{\mathcal{H}} F = m+1+2a$ and the range of the dimension is $(n+1)/2 \le \alpha \le (n+3)/2$.

To conclude the proof, replace $2a = \alpha - m - 1$ in (124) to see that $s(\alpha) = (n-\alpha+m+1)/4$. $\square$

References

[1] An, C., Chu, R., and Pierce, L. B. Counterexamples for high-degree generalizations of the Schrödinger maximal operator. https://arxiv.org/abs/2103.15003.

[2] Barceló, J. A., Bennett, J., Carbery, A., and Rogers, K. M. On the dimension of divergence sets of dispersive equations. *Math. Ann. 349*, 3 (2011), 599–622.

[3] Ben-Artzi, M., and Devinatz, A. Local smoothing and convergence properties of Schrödinger type equations. *J. Funct. Anal. 101*, 2 (1991), 231–254.

[4] Beresnevich, V., and Velani, S. A mass transference principle and the Duffin-Schaeffer conjecture for Hausdorff measures. *Ann. of Math. (2) 164*, 3 (2006), 971–992.

[5] Besicovitch, A. S. Sets of Fractional Dimensions (IV): On Rational Approximation to Real Numbers. *J. London Math. Soc. 9*, 2 (1934), 126–131.

[6] Bourgain, J. On the Schrödinger maximal function in higher dimension. *Tr. Mat. Inst. Steklova 280* (2013), 53–66.

[7] Bourgain, J. A note on the Schrödinger maximal function. *J. Anal. Math. 130* (2016), 393–396.

[8] Carleson, L. Some analytic problems related to statistical mechanics. In *Euclidean harmonic analysis (Proc. Sem., Univ. Maryland, College Park, Md., 1979)* (1980), vol. 779 of *Lecture Notes in Math.*, Springer, Berlin, pp. 5–45.

[9] Cho, C.-H., and Lee, S. Dimension of divergence sets for the pointwise convergence of the Schrödinger equation. *J. Math. Anal. Appl. 411*, 1 (2014), 254–260.

[10] Compaan, E., Lucà, R., and Staffilani, G. Pointwise convergence of the Schrödinger flow. *Int. Math. Res. Not. IMRN*, 1 (2021), 599–650.

[11] Dahlberg, B. E. J., and Kenig, C. E. A note on the almost everywhere behavior of solutions to the Schrödinger equation. In *Harmonic analysis (Minneapolis, Minn., 1981)*, vol. 908 of *Lecture Notes in Math.* Springer, Berlin-New York, 1982, pp. 205–209.

[12] Deligne, P. La conjecture de Weil. I. *Inst. Hautes Études Sci. Publ. Math.*, 43 (1974), 273–307.

[13] Demeter, C., and Guo, S. Schrödinger maximal function estimates via the pseudoconformal transformation. https://arxiv.org/abs/1608.07640.

[14] Du, X., Guth, L., and Li, X. A sharp Schrödinger maximal estimate in $\mathbb{R}^2$. *Ann. of Math. (2) 186*, 2 (2017), 607–640.

[15] Du, X., Guth, L., Li, X., and Zhang, R. Pointwise convergence of Schrödinger solutions and multilinear refined Strichartz estimates. *Forum Math. Sigma 6* (2018), Paper No. e14, 18.

[16] Du, X., and Zhang, R. Sharp L^2 estimates of the Schrödinger maximal function in higher dimensions. *Ann. of Math. (2) 189*, 3 (2019), 837–861.

[17] Eceizabarrena, D., and Lucà, R. Convergence over fractals for the periodic Schrödinger equation. *Anal. PDE, To Appear*.

[18] Falconer, K. *Fractal geometry*, second ed. John Wiley & Sons, Inc., Hoboken, NJ, 2003. Mathematical foundations and applications.

[19] Iwaniec, H., and Kowalski, E. *Analytic number theory*, vol. 53 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[20] Jarník, V. Diophantische approximationen und Hausdorffsches mass. *Mat. Sb. 36*, 3-4 (1929), 371–382.

[21] Jarník, V. Über die simultanen diophantischen Approximationen. *Math. Z. 33*, 1 (1931), 505–543.

[22] Jonsson, A., and Wallin, H. A trace theorem for generalized Besov spaces with three indexes. In *Fourier analysis and approximation theory (Proc. Colloq., Budapest, 1976), Vol. I* (1978), vol. 19 of *Colloq. Math. Soc. János Bolyai*, North-Holland, Amsterdam-New York, pp. 429–449.

[23] Kato, T. On the Cauchy problem for the (generalized) Korteweg-de Vries equation. In *Studies in applied mathematics*, vol. 8 of *Adv. Math. Suppl. Stud.* Academic Press, New York, 1983, pp. 93–128.

[24] Lee, S. On pointwise convergence of the solutions to Schrödinger equations in $\mathbb{R}^2$. *Int. Math. Res. Not.* (2006), Art. ID 32597, 21.

[25] Lee, S., and Rogers, K. M. The Schrödinger equation along curves and the quantum harmonic oscillator. *Adv. Math. 229*, 3 (2012), 1359–1379.

[26] Lucà, R., and Rogers, K. M. Coherence on fractals versus pointwise convergence for the Schrödinger equation. *Comm. Math. Phys. 351*, 1 (2017), 341–359.

[27] Lucà, R., and Rogers, K. M. Average decay of the Fourier transform of measures with applications. *J. Eur. Math. Soc. (JEMS) 21*, 2 (2019), 465–506.

[28] Lucà, R., and Rogers, K. M. A note on pointwise convergence for the Schrödinger equation. *Math. Proc. Cambridge Philos. Soc. 166*, 2 (2019), 209–218.

[29] Lucà, R., and Ponce-Vanegas, F. Convergence over fractals for the Schrödinger equation. https://arxiv.org/abs/2101.02495.

[30] Moyua, A., Vargas, A., and Vega, L. Schrödinger maximal function and restriction properties of the Fourier transform. *Internat. Math. Res. Notices*, 16 (1996), 793–815.

[31] Pierce, L. B. On Bourgain's counterexample for the Schrödinger maximal function. *Q. J. Math. 71*, 4 (2020), 1309–1344.

[32] Rogers, K. M., Vargas, A., and Vega, L. Pointwise convergence of solutions to the nonelliptic Schrödinger equation. *Indiana Univ. Math. J. 55*, 6 (2006), 1893–1906.

[33] Rogers, K. M., and Villarroya, P. Sharp estimates for maximal operators associated to the wave equation. *Ark. Mat. 46*, 1 (2008), 143–151.

[34] Sawin, W. Deligne's theorem on exponential sums. MathOverflow. (https://mathoverflow.net/users/18060/will-sawin) URL: https://mathoverflow.net/q/396428 (version: 2021-06-29).

[35] Sjögren, P., and Sjölin, P. Convergence properties for the time-dependent Schrödinger equation. *Ann. Acad. Sci. Fenn. Ser. A I Math. 14*, 1 (1989), 13–25.

[36] Sjölin, P. Regularity of solutions to the Schrödinger equation. *Duke Math. J. 55*, 3 (1987), 699–715.

[37] Stein, E. M. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, vol. 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[38] Vega, L. Schrödinger equations: pointwise convergence to the initial data. *Proc. Amer. Math. Soc. 102*, 4 (1988), 874–878.

[39] Žubrinić, D. Singular sets of Sobolev functions. *C. R. Math. Acad. Sci. Paris 334*, 7 (2002), 539–544.

[40] Wang, B., and Wu, J. Mass transference principle from rectangles to rectangles in Diophantine approximation. *Math. Ann.* (2021).

[41] Wang, B.-W., Wu, J., and Xu, J. Mass transference principle for limsup sets generated by rectangles. *Math. Proc. Cambridge Philos. Soc. 158*, 3 (2015), 419–437.

[42] Wang, X., and Zhang, C. Pointwise convergence of solutions to the Schrödinger equation on manifolds. *Canad. J. Math. 71*, 4 (2019), 983–995.

[43] Zariski, O., and Samuel, P. *Commutative algebra. Vol. II.* The University Series in Higher Mathematics. D. Van Nostrand Co., Inc., Princeton, N. J.-Toronto-London-New York, 1960.